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ABSTRACT

The disturbance field generated by a point source is obtained for steady, three dimensional, isentropic, magnetohydrodynamic flow. The flow is governed by the Lundquist equations linearized about constant velocity, matter density, and magnetic field. The flow is assumed to be hyperliptic, i.e., the negative of the free stream velocity lies within the fast wave front of the time-dependent characteristic surface but outside the two cusped slow wave fronts of this surface. There are both real and complex characteristics.

New dependent variables are introduced which separate the flow into transverse and compressive components. The transverse component is propagated one-dimensionally with the Alfven velocity. The determination of the compressive component is reduced to finding the fundamental solution of a fourth order homogeneous differential equation with constant coefficients. The solution is required to satisfy smoothness and causality conditions and a condition on the growth at infinity.

The fundamental solution is found by extending the plane wave representation for the fundamental solution of hyperbolic equations to the hyperliptic case. The fundamental solution for a hyperbolic equation is extended to include plane

waves with complex wave speeds. This function will not be continuous across the plane $z = 0$. A solution for $z \neq 0$ of the homogeneous equation is then added so that the resulting function is smooth across $z = 0$, $x^2 + y^2 \neq 0$. This function is shown to be the fundamental solution for the compressive component of the steady flow.

Since there are two directions for which the real wave speeds coincide, it is possible that the solution is singular across the part of the plane ruled by the tangent half lines to the line segment joining the slow wave fronts of the time-dependent characteristic surface. The steady flow is shown to be smooth across this surface. The singularities of the steady flow across the forward-facing nappes of the characteristic cone are computed.

Section 1. Introduction

We treat steady, three dimensional, isentropic, magnetohydrodynamic flow past a point source. The flow is governed by the Lundquist equations [1] which represent a non-dissipative fluid with scalar pressure tensor, infinite conductivity, and in which displacement currents have been ignored. We examine only small steady disturbances imposed on a fluid initially with constant velocity, matter density, and magnetic field; that is, we employ the Lundquist equations linearized about these quantities. It is assumed that the flow is hyperliptic, i.e., the negative of the free stream velocity lies within the fast wave front of the time-dependent characteristic surface but outside the two cusped slow wave fronts of this surface. The equations for steady flow will then have both real and complex characteristics. We shall find the fundamental solution for the steady flow, i.e., the disturbance field generated by a point source. The field quantities must be regarded not as ordinary functions but as distributions. The fundamental solution will have both elliptic properties (e.g. the disturbance will extend throughout space) and hyperbolic properties (e.g. singularities will be carried on the real branches of the characteristic cone).

We introduce new dependent variables, following H. Grad [2], which simplify the analysis by separating

the transverse and compressive components of the flow. The transverse component is propagated one dimensionally with the Alfven velocity and its contribution to the steady flow characteristic cone is two degenerate Alfven cones, i.e. half lines which are the loci of Alfven wave fronts.

The determination of the compressive component is reduced to finding the fundamental solution of a fourth order homogeneous partial differential equation with constant coefficients. The solution is required to satisfy three conditions. The first condition is a causality condition which states that the solution should not be singular on the backward-facing nappes of the characteristic cone. This type of condition is well known in gas dynamics. The second condition is that the solution is required to be smooth in all of space except on the forward-facing nappes of the characteristic cone. Finally, we require the compressive component to vanish at infinity as the inverse square of the distance, i.e. as $\frac{1}{x^2+y^2+z^2}$, except, of course, on the forward-facing nappes. We expect the fundamental solution satisfying these three conditions will be unique; however we do not give a uniqueness proof.

Hyperliptic steady flow has the property that the forward-facing nappes of the characteristic cone lie in a half space, and we choose coordinates so that this is the

space $z > 0$. The causality condition is then that the solution be smooth for $z < 0$. The characteristic polynomial gives two real and two non-real complex conjugate wave speeds in each direction. There are two directions where the real wave speeds coincide.

We find the fundamental solution by extending the plane wave representation for the fundamental solution of hyperbolic equations to the hyperliptic case. If our equation were hyperbolic, i.e. if all the wave speeds were real, the fundamental solution would be $H(z)u_1$, where H is the Heaviside function. The function u_1 consists of the Laplacian (in x and y) iterated three times applied to a superposition of plane wave functions (functions of $x \cos \theta + y \sin \theta + \lambda_i(\theta)z$ where $\lambda_i(\theta)$, $i = 1,2,3,4$ is a wave speed in the θ-direction). The plane wave functions used include $\log |x \cos \theta + y \sin \theta + \lambda_i(\theta)z|$, $i = 1,2,3,4$. We begin modifying u_1 by extending these logarithms into the complex plane for the plane waves with complex wave speeds. The resulting expression, u_2, will be discontinuous across the plane $z = 0$. We correct this by finding a "smoothing" function, M, where M is a solution of the homogeneous equation for $z \neq 0$ and $u_3 = H(z)u_2 + M$ is smooth across the plane $z = 0$, $x^2+y^2 \neq 0$. The function M is found by assuming it to be a superposition of plane waves similar in form to u_2. However, we only include plane waves with complex wave speeds so as to satisfy the causality

condition, i.e. so as not to introduce singularities on the backward-facing nappes of the characteristic cone. The weight functions included in the superposition composing M are determined by the condition that u_3 and its first three normal derivatives be continuous across $z = 0$, $x^2+y^2 \neq 0$. The function u_3 and its first two normal derivatives will then be continuous across $z = 0$ while its third normal derivative has a jump of a constant times $\delta(x)\delta(y)$, the two-dimensional delta function, across this plane. We then easily exploit this property to show that u_3 is a fundamental solution.

The proof that u_3 satisfies the three conditions above is straightforward. The causality condition is satisfied by virtue of our method of selecting M and we show u_3 is (real) analytic for $z < 0$. We use the Cauchy-Kowalewski theorem to show u_3 is analytic in a neighborhood of $z = 0$, $x^2+y^2 \neq 0$. We finally show by a displacement of contour method that u_3 is analytic for $z > 0$ except on the forward-facing nappes of the characteristic cone. The fundamental solution is a function homogeneous of degree one. The compressive component is composed of linear combinations of derivatives of third and fourth order of the fundamental solution and will thus have the desired rate of decay at infinity.

The forward-facing nappes of the characteristic cone consist of the two surfaces ruled by the tangent

half lines from the negative of the free stream velocity to the slow wave fronts of the time-dependent characteristic surface. It also contains the part of the plane ruled by the tangent half lines to the line segment joining the slow wave fronts. This planar segment arises because of the presence of real double points in the characteristic polynomial. The singularity across it is computed and shown to vanish with the application of any derivative tangential to the planar segment. The compressive component is composed of linear combinations of derivatives of the fundamental solution and the fundamental solution is differentiated tangentially at least once in every term of these linear combinations. The compressive component is thus smooth across the planar segment.

For our method of solution, we use coordinate systems which have the forward-facing nappes of the characteristic cone in the space $z > 0$. A Cartesian coordinate system with positive z-axis in the direction of the free stream, such as is often used in steady flow problems, is then suitable only if the forward-facing nappes lie completely downstream. This will occur for equilibria for which the negative of the free stream velocity lies outside the slow normal speed loci. A Cartesian coordinate system with z-axis perpendicular to the unperturbed magnetic field has the forward-facing cone in $z > 0$ for every hyperliptic flow. Various properties of the characteristic polynomial are

derived in these coordinate systems.

The behavior of the fundamental solution near the characteristic cone (excluding the planar segment) has been treated in the hyperbolic case, [3].

The behavior here is similar since the terms of the fundamental solution which become singular in each case have the same form. Our discussion is therefore brief and serves chiefly to show the similarity to the hyperbolic case.

Section 2. Properties of the Lundquist Equations

A: The Lundquist Equations

We treat a perfectly conducting, isentropic fluid described by a velocity field $\vec{u}$, density ρ, and scalar pressure p which is a function of the density alone, i.e., $p = p(\rho)$. The electric field $\vec{E}$ is given by

$$\vec{E} = \vec{B} \times \vec{u}$$

where $\vec{B}$ is the magnetic field, since the fluid is a perfect conductor. We ignore the displacement current in Maxwell's equations, so

$$\mu \vec{J} = \nabla \times \vec{B} ,$$

where $\vec{J}$ is the current per unit area and μ is the specific inductive capacity. The remaining equation of Maxwell's equations is

$$\partial \vec{B}/\partial t + \nabla \times (\vec{B} \times \vec{u}) = 0 , \tag{1A}$$

where we have used $\vec{E} = \vec{B} \times \vec{u}$. Equation (1A) implies that $\nabla \cdot \vec{B} = 0$ for all time provided it is zero initially.

We neglect dissipative effects such as heat conduction and viscosity. The equations of conservation of mass and momentum are then

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\ (\rho \vec{u}) = 0 \tag{1B}$$

and

$$(1c) \qquad \rho \frac{\partial \vec{u}}{\partial t} + \rho(\vec{u}\cdot\nabla)\vec{u} + a^2\nabla\rho + \frac{1}{\mu}\vec{B}\times(\nabla\times\vec{B}) = 0$$

respectively, where $a = \sqrt{dp/d\rho}$ is the speed of sound and $-\frac{1}{\mu}\vec{B}\times(\nabla\times\vec{B})$ is the Lorentz force per unit volume.

Equations (1) are the Lundquist equations for an isentropic fluid and form a nonlinear, first order, Galilean invariant system for the functions $\vec{u}$, $\vec{B}$ and ρ. An equilibrium solution to this system is given by $\vec{u} = \vec{u}_0$, $\vec{B} = \vec{B}_0$, and $\rho = \rho_0$, where $\vec{u}_0$, $\vec{B}_0$ and ρ_0 are any set of constants. We consider a subset of these constant equilibria which we will describe later in this section. We linearize about any equilibrium taken from this subset and look for steady flows past a point source.

B. The Characteristic Cone of the Lundquist Equations

The subset of equilibria we consider may be described geometrically with the aid of the characteristic cone. This cone is the set of surfaces of singularity, or wave fronts, which propagate from a point source in the initial plane $t = 0$. We shall also use this cone to construct the singularity surfaces for the steady flow and to determine the proper causality condition needed to complete the mathematical formulation of the steady flow problem. The characteristic equation of equations (1) (the equation for the characteristic surfaces) will also be useful. We give a brief summary of relevant results on the characteristic

equation and cone using [4, 5, 6] and then describe the equilibria we consider.

A characteristic surface $\phi(x,y,z,t) = 0$ for a first order system and for a given solution thereof is a surface across which the solution is continuous while its normal derivative (i.e. the normal derivative of each component of the solution) is singular. We consider a point on the characteristic surface where the solution has the value $\vec{u} = \vec{u}_1$, $\vec{B} = \vec{B}_1$ and $\rho = \rho_1$. We use equations (1) to find a linear algebraic system of equations for the normal derivative of the solution at the point in terms of $\vec{u}_1$, $\vec{B}_1$, and ρ_1. This system may be found by introducing a coordinate system near the given point with one set of coordinate surfaces given by $\phi(x,y,z,t)$ = constant, and the other coordinates varying in each such surface. The normal derivative of the solution is then $\vec{u}_\phi$, $\vec{B}_\phi$, and ρ_ϕ. This normal derivative will be continuous across $\phi(x,y,z,t)=0$ unless the determinant of the algebraic system vanishes. The determinant is a homogeneous polynomial, $P(\phi_t,\phi_x,\phi_y,\phi_z)$, in the components of the normal vector, $(\phi_t,\phi_x,\phi_y,\phi_z)$, to the characteristic surface at the point. The coefficients of $P(\phi_t,\phi_x,\phi_y,\phi_z)$ depend on $\vec{u}_1$, $\vec{B}_1$ and ρ_1, but not on the coordinates of the point, i.e. not on the independent variables (x,y,z,t), since the equations (1) do not depend explicitly on the independent variables. Setting the determinant to zero, we get the characteristic equation,

$P(\phi_t,\phi_x,\phi_y,\phi_z) = 0$, which is a partial differential equation for the characteristic surface with coefficients depending upon the solution to equations (1).

If we linearize equations (1) about a constant equilibrium, $\vec{u} = \vec{u}_0$, $\vec{B} = \vec{B}_0$, and $\rho = \rho_0$, and find the characteristic equation of the resulting linear system, this equation will coincide with that for the nonlinear system (equations (1)) if we set $\vec{u}_1 = \vec{u}_0$, $\vec{B}_1 = \vec{B}_0$, and $\rho_1 = \rho_0$. Moreover, the characteristic surface will be independent of the solution considered, since the system is linear. The characteristic equation for the linear system is a first order, homogeneous, partial differential equation for the characteristic surface, $\phi(x,y,z,t) = 0$, with constant coefficients which depend upon the equilibrium constants $\vec{u}_0$, $\vec{B}_0$, and ρ_0. The characteristic surfaces for linearized steady flow may be found by seeking solutions of the above characteristic equation with $\phi_t \equiv 0$.

The characteristic equation for equations (1), which have been linearized about the constant equilibrium $\vec{u} = \vec{u}_0$, $\vec{B} = \vec{B}_0$, and $\rho = \rho_0$, may be found by first writing equations (1) in matrix form as

$$C_0\vec{v}_t + \sum_{i=1}^{3} C_i\vec{v}_{x_i} = 0 \ , \quad \vec{v} = \begin{pmatrix} \vec{u} \\ \vec{B} \\ \rho \end{pmatrix},$$

where the matrices C_i, $i = 0,1,2,3$, are evaluated at $\vec{v}_0 = \begin{pmatrix} \vec{u}_0 \\ \vec{B}_0 \\ \rho_0 \end{pmatrix}$, and then changing coordinates as mentioned before. The characteristic equation is

$$\det\left[C_0\phi_t + \sum_{i=1}^{3} C_i\phi_{x_i}\right] = 0$$

which, after a computation, is

$$(2)\quad P(\phi_t,\phi_x,\phi_y,\phi_z) = [\phi'^2 - (\vec{A}_0\cdot\nabla\phi)^2][(\phi')^4-(a_0^2+A_0^2)(\phi'^2)$$

$$\cdot(\nabla\phi)^2 + a_0^2(\vec{A}_0\cdot\nabla\phi)^2\ (\nabla\phi)^2] = 0$$

where $\phi' = \phi_t + \vec{u}_0\cdot\nabla\phi$, $a_0 = \dfrac{dp(\rho_0)}{d\rho}$ is the sound speed,

$\vec{A}_0 = \dfrac{\vec{B}_0}{\sqrt{\mu\rho_0}}$ is the Alfven velocity,

$u_0 = |\vec{u}_0|$, and $A_0 = |\vec{A}_0|$.

The characteristic surfaces for steady linearized flow satisfy

$$(3)\quad P(0,\phi_x,\phi_y,\phi_z) = [(\vec{u}_0\cdot\nabla\phi)^2 - (\vec{A}_0\cdot\nabla\phi)^2]$$

$$\cdot\begin{bmatrix}(\vec{u}_0\cdot\nabla\phi)^4-(a_0^2+A_0^2)(\vec{u}_0\cdot\nabla\phi)^2(\nabla\phi)^2\\ + a_0^2(\vec{A}_0\cdot\nabla\phi)^2(\nabla\phi)^2\end{bmatrix} = 0$$

The first component in equations (2) and (3) corresponds to the Alfven or transverse disturbances and the second to the compressive disturbances.

The characteristic cone for equations (1) linearized about $\vec{u} = \vec{u}_0$, $\vec{B} = \vec{B}_0$, and $\rho = \rho_0$, at a given point is given by the envelope of all the characteristic planes through the point. (Its equation will be independent of the point considered, since the characteristic equation has this property.) A plane through the origin at $t = 0$ has the equation

$$- ct + \vec{x} \cdot \vec{n} = 0 \ , \quad \vec{x} = (x,y,z), \quad \vec{n} = (n_x,n_y,n_z), \ n_x^2+n_y^2+n_z^2 = 1,$$

and will be characteristic if $(-c,\vec{n})$ is a root of

$P(c,n_x,n_y,n_z)$, the characteristic polynomial.

The six roots $-c_i(\vec{n})$, $i = 1,\ldots,6$, may be shown to be real for all $\vec{n}$ and equations (1) are hyperbolic. The envelope of a family of characteristic surfaces is again a characteristic surface, since the characteristic equation is of first order. The characteristic cone, which is the envelope of the above planes, consists of several sheets which give the singularity surfaces for a point source at the origin at $t = 0$. This cone, or more specifically, its convex hull, determines the maximum possible range of influence of the point source for the linear system [7]. The characteristic surface is defined as the intersection of the characteristic cone with the plane $t = 1$. This locus is three dimensional with rotational symmetry about the axis of $\vec{A}_0$, and a section containing $\vec{A}_0$ is given in Figure 1 for the case $A_0 < a_0$.

The characteristic surface when $A_0 > a_0$ is the same, except the transverse disturbances now lie on the fast locus with $\vec{A}_0 = \vec{OQ}$. When $A_0 = a_0$, the two inner, or slow, loci join with the outer, or fast, locus.

We note that the line segment, $\overline{R'R}$, joining the inner loci must be regarded, a priori, as part of the characteristic

surface, although it is not given by the envelope construction above. The reason it must be included will be given later when we discuss the effect of double points, which are values of $\vec{n}$ for which two or more $c_i(\vec{n})$ coincide. This line segment may, however, be shown not to be a singularity surface and thus not part of the characteristic surface. We will use the characteristic cone for equations (1) to construct the characteristic cone for the steady flow problem, and show, by examining the explicit solution, that the line segment $\overline{R'R}$ may be neglected in this construction.

We now describe the subset of equilibria treated. Consider the vector $-\vec{u}_0$ from the origin at $t = 1$ and look at the section of the characteristic surface containing both $-\vec{u}_0$ and $\vec{A}_0$. We denote the set of all points outside the outer locus by region I, those inside the inner loci by region II, and those in between the outer locus and the inner loci by region III. We will treat the subset of equilibria for which $-\vec{u}_0$ lies in region III. The significance of this choice will be discussed when we consider the characteristic cone for the steady flow problem in Section 3.

Section 3. The Steady Flow Equations

A. Derivation

We introduce new dependent variables, as used by H. Grad [2], which will simplify our discussion by separating the equations into two independent systems: one will have as its characteristic equation the first component in equation (3) set to zero, and the other will have the second component set to zero. We thereby avoid having to deal with a system which has a reducible characteristic equation. We note that with a nontrivial geometry, the boundary conditions will, in general, couple these systems.

We linearize the Lundquist equations (equations (1)) about a constant equilibrium $(\vec{B}_0, \vec{u}_0, \rho_0)$ and introduce dimensionless quantities $(\vec{B}, \vec{u}, \rho)$ for the perturbed quantities $(\vec{B}', \vec{u}', \rho')$ by setting

$$\vec{B} = \frac{\vec{B}'}{|B_0|}\,, \quad \vec{u} = \frac{\vec{u}'}{u_0}\,, \quad \rho = \frac{\rho'}{\rho_0}\,.$$

The motion of the fluid past a source, $(\rho_0\rho_s(\vec{x}), \rho_0 u_0 \vec{M}_s(\vec{x}), B_0\,\vec{B}_s(\vec{x}))$, is given by the following equations:

$$u_0 \nabla\cdot\vec{u} + \vec{u}_0\cdot\nabla\rho = \rho_s \tag{4A}$$

$$u_0(\vec{u}_0\cdot\nabla)\vec{u} + a_0^2\nabla\rho + A_0\vec{A}_0 \times(\nabla\times\vec{B}) = u_0\vec{M}_s \tag{4B}$$

$$u_0\nabla\times(\vec{A}_0\times\vec{u}) + A_0(\vec{u}_0\cdot\nabla)\vec{B} = A_0\vec{B}_s \quad , \tag{4C}$$

where $\rho_s(\vec{x})$, $\vec{M}_s(\vec{x})$, $\vec{B}_s(\vec{x})$ are C_0^∞ functions, i.e. they are infinitely differentiable with compact support, and $\vec{B}_s(\vec{x})$ satisfies $\nabla\cdot\vec{B}_s(\vec{x}) = 0$. We also have the equation $\nabla\cdot\vec{B} = 0$, which is a boundary condition at infinity. If $\nabla\cdot\vec{B} = 0$ at infinity, then, by taking the divergence of equation (4C), we find $\nabla\cdot\vec{B} = 0$ everywhere.

It would be tempting, in order to find the fundamental solution of equations (4), to consider the flow past a point source, i.e. to replace the source terms ρ_s, $\vec{M}_s$, and $\vec{B}_s$ in equations (4) by the functions $\rho_c\delta(\vec{x})$, $\vec{M}_c\delta(\vec{x})$, and $\vec{B}_c\delta(\vec{x})$, where ρ_c, $\vec{M}_c$, and $\vec{B}_c$ are constants. However, we would then have a contradiction, since, from equation (4C),

$$\nabla\cdot(u_0\nabla\times(\vec{A}_0\times\vec{u})+A_0(\vec{u}_0\cdot\nabla)\vec{B}) = 0 \neq A_0\nabla\cdot\vec{B}_c\delta(\vec{x}) \ ,$$

unless $\vec{B}_c = 0$. We will therefore proceed by showing that every component, v_i, of the solution to equations (4) satisfies the equation $Lv_i = q_i$, where L is the same differential operator for all i and q_i is a linear combination of various derivatives of the given source functions. We then solve the equation

$$Lv = \delta^\alpha(x)\ \delta^\beta(y)\ \delta^\gamma(z) \ , \quad \text{where } \delta^\alpha(x_i) = \frac{d^{\alpha-1}\delta(x_i)}{dx_i^{\alpha-1}} \ ,$$

and the solution to equations (4) will follow by forming the proper superpositions. (An alternate approach for

resolving the above difficulty is given by H. Weitzner [8].)

B. The Compressive Component

We start by introducing the new dependent variables (α,β,γ), following H. Grad [2], with $\alpha = \nabla\cdot\vec{u}$, $\beta = \vec{A}_0\cdot\vec{u}$ and $\gamma = \vec{A}_0\cdot\vec{B}$. These three variables and the density ρ satisfy the following equations. Equation (4A) gives directly

$$u_0\alpha + \vec{u}_0\cdot\nabla\rho = \rho_s \ . \tag{5A}$$

The divergence of equation (4B) gives

$$u_0(\vec{u}_0\cdot\nabla)\alpha + a_0^2\Delta\rho + A_0\,\Delta\gamma = u_0\nabla\cdot\vec{M}_s \tag{5B}$$

The inner product of $\vec{A}_0$ with equations (4B) and (4C) gives

$$u_0\vec{u}_0\cdot\nabla\beta + a_0^2\vec{A}_0\cdot\nabla\rho = u_0\vec{A}_0\cdot\vec{M}_s \tag{5C}$$

and

$$u_0A_0^2\alpha - u_0\vec{A}_0\cdot\nabla\beta + A_0\vec{u}_0\cdot\nabla\gamma = A_0\vec{A}_0\cdot\vec{B}_s \ . \tag{5D}$$

These equations are equivalent to a single fourth order equation for each dependent variable of the form $Lv_i = q_i$, as indicated above. These equations may be found by first writing equations (5) in matrix form as

$$A_1\vec{v}_x + A_2\vec{v}_y + A_3\vec{v}_z = \vec{R}$$

where A_i are constant matrices, $\vec{v} = \begin{pmatrix} \rho \\ \alpha \\ \beta \\ \gamma \end{pmatrix}$, and

$$\vec{R} = \begin{pmatrix} \rho_s \\ u_0 \nabla \cdot \vec{M}_s \\ u_0 \vec{A}_0 \cdot \vec{M}_s \\ A_0 \vec{A}_0 \cdot \vec{M}_s \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} .$$

We may find the equation satisfied by each component in a systematic way by formally solving the algebraic system

$$(A_1 D_x + A_2 D_y + A_3 D_z)\vec{v} = \vec{R} ,$$

where the differential operators D_x, D_y, D_z are treated as constants [7]. The solution is

$$\det (A_1 D_x + A_2 D_y + A_3 D_z)\vec{v} = (A_1 D_x + A_2 D_y + A_3 D_z)^* \vec{R}$$

where the matrix of the transposed cofactors of the elements of a matrix B is denoted by B^*. Each component of $\vec{v}$ then satisfies the same equation but with a different forcing term, namely:

$$(6) \quad \det (A_1 D_x + A_2 D_y + A_3 D_z) v_i = \sum_{j=1}^{4} \sum_{\ell+m+n=3}^{4} b^{ij}_{\ell mn} D_{\ell mn} R_j$$

$$i = 1,2,3,4$$

where $b^{ij}_{\ell mn}$ are constants , and $D_{\ell mn}$ is the differential operator $\dfrac{\partial^{\ell+m+n}}{\partial x^{\ell} \partial y^{m} \partial z^{n}}$.

In other words, the right hand side of equation (6) consists of a linear combination of the partial derivatives

of the R_i, $i = 1,2,3,4$, where the degree of each term is three or four. We will explicitly give some of the forcing terms when we discuss the singularity surfaces of the solution in Section 6-B.

It will then be sufficient to solve the equation

$$(7) \quad \det\,(A_1 D_x + A_2 D_y + A_3 D_z)v = \delta^{\alpha}(x)\ \delta^{\beta}(y)\ \ \delta^{\gamma}(z)$$

where $\alpha + \beta + \gamma = 6$ or 7.
We then obtain the solution to equations (5) by forming the proper superpositions of these solutions, using various values of α, β and γ.

The characteristic equation for equations (7) (their characteristic equations are identical since they do not depend on the forcing term) is the same as that for equations (5). (The characteristic equation for a single higher order equation is defined in essentially the same way as for a system.) Moreover, it may be obtained from equation (3) by setting the compressive component to zero.

C. The Transverse Component

We obtain two additional equations for the new dependent variables $j = \vec{A}_0 \cdot (\nabla \times \vec{B})$ and $\omega = \vec{A}_0 \cdot (\nabla \times \vec{u})$, again following H. Grad [2]. These equations are found by applying $\vec{A}_0 \cdot$curl to equations (4B) and (4C). The results

are, denoting the directional derivatives $A_0^{-1}\vec{A}_0\cdot\nabla$ and $u_0^{-1}\vec{u}_0\cdot\nabla$ by d/dp and d/dq respectively,

$$(8A) \qquad u_0^2 \frac{d\omega}{dq} - A_0^2 \frac{dj}{dp} = u_0\vec{A}_0\cdot(\nabla\times\vec{M}_s)$$

and

$$(8B) \qquad u_0 \frac{d\omega}{dp} - u_0 \frac{dj}{dq} = \vec{A}_0\cdot(\nabla\times\vec{B}_s) .$$

Combining these equations, we find that both j and ω satisfy one-dimensional wave equations with wave speed A_0/u_0 (the Alfven speed A_0 in our original coordinates), namely:

$$u_0^2 \frac{d^2\omega}{dq^2} - A_0^2 \frac{d^2\omega}{dp^2} = u_0 \frac{d}{dq}(\vec{A}_0\cdot\nabla\times\vec{M}_s) - \frac{A_0^2}{u_0}\frac{d}{dp}(\vec{A}_0\cdot\nabla\times\vec{B}_s)$$

$$u_0^2 \frac{d^2 j}{dq^2} - A_0^2 \frac{d^2 j}{dp^2} = u_0 \frac{d}{dp}(\vec{A}_0\cdot\nabla\times\vec{M}_s) - u_0\frac{d}{dq}(\vec{A}_0\cdot\nabla\times\vec{B}_s) .$$

The characteristic equation for the system (8) is that obtained by setting the transverse factor in equation (3) to zero. The solution of the above wave equations is well known, and we shall be concerned in the sequel only with equations (5).

D. Inverting the Solution

After the solution of equations (5) and (8), there remains the problem of solving for $\vec{u}$ and $\vec{B}$ given α, β, γ, j, and ω (which are $\nabla\cdot\vec{u}$, $\vec{A}_0\cdot\vec{u}$, $\vec{A}_0\cdot\vec{B}$, $\vec{A}_0\cdot(\nabla\times\vec{B})$,

and $\vec{A}_0 \cdot (\nabla \times \vec{u})$, respectively). Since the components of $\vec{u}$ and $\vec{B}$ along $\vec{A}_0$ are known, we have four differential equations (three from inverting α, j, and ω, and the equation div $\vec{B} = 0$) to determine these functions on the plane orthogonal to $\vec{A}_0$. H. Grad [2] showed that these equations are invertible if a boundedness condition is imposed at infinity and if the inhomogeneous terms, α, β, γ, j, and ω are sufficiently smooth. In our case, we may directly invert the appropriate linear combinations of the solutions of equation (7) without the necessity of first obtaining smooth source functions, since these solutions will be given as various derivatives applied to smooth functions. We may first invert these smooth functions and then apply the derivatives. This procedure is correct since derivatives of distributions may always be interchanged.

A full discussion of the implications of these global resolutions (with α, β, γ, j, and ω smooth) is given by H. Grad [2].

E. Construction of the Steady Flow Characteristic Cone

We may use the characteristic surface for equations (1) to give a geometric construction of the characteristic cone for equations (5) [6]. To completely determine the solution of equations (5), it is necessary to add a causality requirement. This may be obtained by considering

the steady flow to be the limit in infinite time, of a time dependent flow. Equivalently, since equations (1) are Galilean invariant, we may regard the steady flow past a point source as the result of the source moving with velocity $-\vec{u}_0$ from infinity into a stationary fluid. At each point in time, the source causes a wave pattern, as shown in Figure 1, to propagate. The possible singularity surfaces for the steady flow are given by the envelope of these waves for $t < t_0$. This gives a single-napped (forward-facing) cone while equations (5) will have the full cone (both nappes) as its characteristic cone, i.e. as its set of possible singularity surfaces. We therefore impose the causality condition that the solutions we seek be smooth across the backward-facing nappes. The forward-facing cone may be constructed by drawing the vector $-\vec{u}_0$ from the origin at $t = 1$ and then the forward-facing cone is ruled by the tangent half lines from $-\vec{u}_0$ to the characteristic surface.

There are three qualitatively different cases depending on whether $-\vec{u}_0$ lies in region I, II or III. Figure 2 shows the plane containing $-\vec{u}_0$ and $\vec{A}_0$, with $-\vec{u}_0$ lying in region III. For our case, $-\vec{u}_0$ lies in region III and the construction gives a cone (single-napped) which is composed of two branches ruled by the tangent half lines from $-\vec{u}_0$ to the two cusped slow loci of the characteristic surface of equations (1), as well as an additional planar

segment composed of tangents to the line joining the slow loci. As noted above, we shall show that this planar segment is not a singularity surface for the steady flow.

The characteristic cone may also be found by forming the envelope of all characteristic planes through the point source, and then using the causality condition to choose the proper nappes. We set $\vec{n} = (\cos\theta, \sin\theta)$ and $\vec{x} = (x,y)$. A plane with equation $\vec{x}\cdot\vec{n} - \lambda z = 0$ is characteristic if $-\lambda(\theta)$ is a root of the characteristic polynomial of equations (7). The characteristic polynomial is

$$
\begin{aligned}
Q(\cos\theta,\sin\theta,\lambda) &= [\vec{u}_0\cdot(\cos\theta,\sin\theta,\lambda)]^4 \\
(9) \qquad &- (a_0^2+A_0^2)[\vec{u}_0\cdot(\cos\theta,\sin\theta,\lambda)]^2(1+\lambda^2) \\
&+ a_0^2\ (\vec{A}_0\cdot(\cos\theta,\sin\theta,\lambda))^2\ (1+\lambda^2)\ .
\end{aligned}
$$

Each real root $-\lambda(\theta)$ $(0 \leq \theta \leq 2\pi)$ of this polynomial gives rise to a branch of the cone. We would then expect the polynomial to have two real and two nonreal roots in region III. We shall in fact be able to prove this algebraically for a large part of region III, for two specified coordinate systems. The planar segment of the cone is not given by the envelope construction but arises due to the presence of real double points in the characteristic polynomial, i.e. points where the real roots coincide.

This will be discussed when we analyze the solution in Section 6-A.

The method used to solve equations (7), and hence equations (5), requires the existence of an open half space containing the forward-facing single-napped cone, and we introduce coordinates so that this is the space $z > 0$. We can (nonuniquely) accomplish this for any point in region III as follows: through the origin of the steady flow coordinate system (which in Fig. 2 is at $-\vec{u}_0$) we take the positive z-axis of a Cartesian coordinate system as the half line perpendicular to the direction of $\vec{A}_0$ and intersecting the line containing $\vec{A}_0$. All half lines tangent to the time dependent characteristic surface, and hence the single-napped cone, will lie in $z > 0$. We note that this property even holds for those equilibria in region III which have part of their forward-facing cone upstream. This property also holds in region I, but never in region II.

The z-axis for our case ($-\vec{u}_0$ in region III) is somewhat analogous to a time axis for a non-steady problem, in that the discontinuities of the solution propagate with increasing z. The analogy is not complete in that we expect the solution to extend throughout space, due to the existence of nonreal roots of the characteristic polynomial. The cone then does not give the range of influence for the source. The mathematical formulation of our problem has thus both hyperbolic and elliptic behavior, and the flow is called hyperliptic.

The case of $-\vec{u}_0$ lying in region I has been treated elsewhere [9]. The forward-facing nappes lie in a half space and (in a coordinate system with these nappes in $z > 0$) the characteristic polynomial has all real roots. The forward-facing nappes are composed of three branches: an outer branch, ruled by half lines to the fast locus, and a branch to each of the slow loci. Here, as in our case, the solution is smooth across the planar segment ruled by the tangent half lines to the line connecting the slow loci. The mathematical formulation of the problem in region I leads to a Cauchy problem for a hyperbolic equation, and the z-axis is completely analogous to a time axis. In particular, the convex hull of the characteristic cone gives the maximum range of influence of the point source.

For $-\vec{u}_0$ in regionII, the forward-facing cone consists of two branches; however, it does not lie completely in a half space. The mathematical formulation will thus differ from that in regions I and III, and this problem remains to be treated.

F. A Note on Uniqueness of the Steady Flow

We close this section by noting that we do not prove that the steady flow solution obtained is unique, although with the causality condition, as well as the growth at infinity and smoothness conditions given in Section 5-A, uniqueness seems likely. One difficulty is that the nature

of the characteristic polynomial (equation (9)) may be qualitatively different in different coordinate systems. For example, if a characteristic plane is chosen to be the plane $z = 0$, we find an infinite root, i.e. the polynomial will be of lower order for the corresponding value of θ.

We shall assume that the causality condition together with the smoothness and growth conditions to be given (Section 5-A) are sufficient to insure the uniqueness of the solution of equation (7) and hence of equations (5). We may therefore use any appropriate coordinate system to facilitate the analysis. We consider two such systems in the following section, and derive properties of the characteristic polynomial in each of these systems.

Section 4. The Characteristic Polynomial

We have reduced the problem of finding the steady flow to that of finding a solution to equation (7) which satisfies various conditions to be described in Section 5-A. In this section, we derive certain properties of the characteristic polynomial, which will be used later to derive and analyze the steady flow. We use the coordinate system given in Section 3-E which has the forward-facing cone in $z > 0$ for all equilibria in region III. A second coordinate system, commonly used in gas dynamics, is also discussed. This coordinate system has the forward-facing cone in $z > 0$ for a subregion (III-A) of region III.

We conjecture that the following properties of the characteristic polynomial are true in any coordinate system which contains the forward-facing cone in the space $z > 0$. The characteristic polynomial $Q(\cos\theta, \sin\theta, \lambda)$ (equation (9)) will have two real and two non-real (complex conjugate) roots for every real θ. Moreover, there will be only two double points, $\theta = \theta_0$ and $\theta = \theta_0 + \pi$, $0 \leq \theta \leq 2\pi$, where the real roots coincide. We expect that there are always two real roots since the characteristic cone has two real branches. If there were more than two real roots on some θ interval, these two branches would be doubly covered for those values of θ, while essentially the same branches for $-\vec{u}_0$ in region I are singly covered. Finally, we shall see that a real double

point θ_0, $0 \leq \theta_0 < \pi$, will cause a planar segment of the characteristic cone to appear. However, our construction only gives one such segment, so we believe there is only one real double point.

We need only consider the polynomial for $0 \leq \theta < \pi$, for if $\lambda(\theta)$ is a root, we immediately have $-\lambda(\theta)$ and $-\overline{\lambda(\theta)}$ are roots at the point $\theta + \pi$. We choose the latter to give

$$(10) \qquad \lambda(\theta + \pi) = - \overline{\lambda(\theta)} .$$

The conjugate is chosen since the complex roots never cross the real axis. We remark that our method of solution is easily carried over to the case of any finite number of real double points.

We now consider the coordinate system described in Section 3-E, and we additionally specify $\vec{A}_0 = A_0(0,1,0)$ and $\vec{u}_0 = u_0(0, \cos\phi, \sin\phi)$. The characteristic polynomial (equation (9)) is then

$$(11) \qquad \begin{aligned} Q(\cos\theta, \sin\theta, \lambda) &= u_0^4(\cos\phi \sin\theta + \lambda \sin\phi)^4 \\ &+ u_0^4(1+\lambda^2)\left[\frac{\sin^2\theta}{m^2M^2} - \left(\frac{1}{m^2} + \frac{1}{M^2}\right)(\cos\phi \sin\theta + \lambda \sin\phi)^2 \right] \end{aligned}$$

where $m = u_0/a_0$ and $M = u_0/A_0$ are the Mach numbers.

We first show that Q always has two real roots. Figure 1 shows that a necessary condition for $-\vec{u}_0$ to lie in region III is

$$(12)\qquad u_0 \leq \sqrt{a_0^2 + A_0^2}$$

or

$$\frac{1}{m^2} + \frac{1}{M^2} \geq 1 .$$

We will not consider the degenerate case of parallel flow where $\sin \Phi = 0$ and the characteristic polynomial is of second order.

The coefficient of λ^4 in equation (11) is

$$u_0^4 \sin^2\Phi \,(\sin^2\Phi - (\frac{1}{m^2} + \frac{1}{M^2})) ,$$

which is negative, by inequality (12), so Q is always of fourth order.

Inequality (12) further implies that Q will always be negative at $\lambda = \pm\infty$, but since Q is nonnegative when $\lambda = - \cot \Phi \sin \theta$, where $Q(\cos \theta, \sin \theta, -\cot \Phi \sin \theta) = \frac{u_0^4 \sin^2\theta}{m^2 M^2} (1+\cot^2\Phi \sin^2\theta)$, Q always has two real roots. Moreover, this shows that at any point where Q has two real and two nonreal roots, the real roots can only coincide when $Q(\cos \theta, \sin \theta, - \cot \Phi \sin \theta) = 0$. This can only occur for $\sin \theta = 0$, where, as we shall see below, there is a double point with both real roots vanishing.

A proof that there are two complex roots in region III would be more complicated, since these roots are real for some equilibria in the other regions. For example, they are real, in this coordinate system, provided $-\vec{u}_0$ lies in the subdomain of region I given by $|z| > \sqrt{a_0^2+A_0^2}$ (which implies $|\sin \Phi| > \sqrt{\frac{1}{m^2} + \frac{1}{M^2}}$). We shall instead verify the existence

of two nonreal roots in several limiting regions.

We begin with the case $\sin\theta = 0$ for all points in region III, since this value of θ will not be covered in our various limiting regions (because of the inversion of limits). Without loss of generality, we take $0 \leq \theta < \pi$ and $0 < \phi < 2\pi$, $\phi \neq \pi$. The roots for $\sin\theta = 0$ may be found exactly, since the polynomial is then

$$Q(1,0,\lambda) = \lambda^4 u_0^4 \sin^4\phi - u_0^4\left(\frac{1}{m^2} + \frac{1}{M^2}\right)(1+\lambda^2)\lambda^2 \sin^2\phi$$

with roots

$$\lambda^2 = 0 \quad \text{and}$$

$$\lambda^2 = \frac{1/m^2 + 1/M^2}{\left(\sin^2\phi - \left(\frac{1}{m^2} + \frac{1}{M^2}\right)\right)} < 0$$

by inequality (12), so our condition is verified here, i.e. two roots are real and two complex. We also have exhibited the double real root $\lambda = 0$ for $\sin\theta = 0$. Moreover, there must be two nonreal roots in a neighborhood of $\sin\theta = 0$, by the continuity of the roots in $\sin\theta$ and since the complex roots must coincide to become real. The other two roots must then be real for $\sin\theta$ small, since two real roots always exist. In our remaining calculations, $\sin\theta$ will always be bounded away from zero.

We now consider the case of perpendicular flow, i.e. $\cos\phi = 0$. Here the roots may again be found explicitly. In this case, the polynomial is

$$Q(\cos\theta, \sin\theta, \lambda) = u_0^4\left(1-\left(\frac{1}{m^2}+\frac{1}{M^2}\right)\right)\lambda^4 + u_0^4\left(\frac{\sin^2\theta}{m^2M^2} - \left(\frac{1}{m^2}+\frac{1}{M^2}\right)\right)\lambda^2 + \frac{u_0^4\sin^2\theta}{m^2M^2}$$

which is second order in λ^2. It is negative for $\lambda^2 = \pm\infty$ and positive for $\lambda^2 = 0$ so we have one positive and one negative root λ^2, which gives two real and two nonreal roots. This result also holds in some neighborhood of $\cos\Phi = 0$, by the previous argument.

We next consider the region near parallel flow, i.e. $\sin\Phi = 0$. There are always two real roots for $\sin\Phi \neq 0$, and it is interesting to investigate their behavior at $\sin\Phi \to 0$. Expanding the roots for $\sin\Phi$ small, we find two real roots which behave to leading order like $1/\sin\Phi$. This follows since, for $\lambda = O(1/\sin\Phi)$, the leading order term satisfies

$$\lambda^2\left[\frac{\sin^2\theta}{m^2M^2} - \left(\frac{1}{m^2}+\frac{1}{M^2}\right)(\cos\Phi\sin\theta + \lambda\sin\Phi)^2\right] = 0.$$

We discard the zero roots - the two remaining roots may be trivially found and are real and distinct.

The remaining roots for $\sin\Phi$ small are of zero order in $\sin\Phi$ and, by continuity, will behave like the two roots for parallel flow. These, as is well known, are complex in region III, so our result is true near parallel flow.

We may also show this result for u_0 small (i.e. for $m, M << 1$) by the same methods. We again find that two real roots become infinite and, to leading order, the two zero

order roots satisfy $\sin^2\theta(1+\lambda^2) = 0$, which gives the result here.

As noted before, the only double point for an equilibrium where Q always has two real and two nonreal roots is at $\sin\theta = 0$, where the real roots vanish. The corresponding characteristic plane is $x = 0$, which is the plane of $\vec{u}_0$ and $\vec{A}_0$.

Another useful coordinate system has its z-axis in the $\vec{u}_0$ direction. We take $\vec{A}_0 = A_0(0,\sin\Phi,\cos\Phi)$ and $\vec{u}_0 = u_0(0,0,1)$. The angle Φ is then the angle between $\vec{u}_0$ and $\vec{A}_0$ (in this coordinate system as well as in the one above). The characteristic polynomial (equation (9)) in this coordinate system is

$$\frac{Q(\cos\theta,\sin\theta,\lambda)}{u_0^4} = \lambda^4 - \left(\frac{1}{m^2}+\frac{1}{M^2}\right)\lambda^2(1+\lambda^2) + \frac{1}{m^2M^2}(1+\lambda^2)(\sin\Phi\sin\theta+\lambda\cos\Phi)^2 . \tag{13}$$

This coordinate system has the disadvantage that only points in a subregion, region III-A, of region III will have the forward-facing nappes of their characteristic cone in the space $z > 0$. In the plane of $\vec{u}_0$ and $\vec{A}_0$, the boundary of region III-A is given by those equilibria which have their y-axis tangent to the slow loci of the characteristic surface. An equilibrium will then lie on the boundary of region III-A if $-\vec{u}_0$ is the closest point to the origin on any line tangent to the slow loci. The boundary is then the pedal curve of the slow loci. (The above condition is the

definition of a pedal curve.) This pedal curve is, however, just the slow branches of the normal speed locus, which is the polar plot of the slow compressive wave speeds in each direction. This locus is well known and is shown, along with the fast branch, in Figure 3, where it is superposed on the characteristic surface. In fact, the characteristic surface is usually found as the envelope of all lines which intersect the normal speed locus at their closest distance to the origin. This is the inverse of the construction which gives the normal speed locus as the pedal curve of the characteristic loci. The boundary may also be found analytically, and consists of those equilibria for which the characteristic polynomial is of lower order, i.e.

$$(1 - (\frac{1}{m^2} + \frac{1}{M^2}) + \frac{\cos^2\phi}{m^2M^2}) = 0 . \tag{14}$$

The region III-A now consists of those points in region III which lie outside the two branches of the slow normal speed locus. These points are precisely those whose single-napped cone lies completely downstream. (Downstream in this coordinate system is just the space $z > 0$, and upstream is $z < 0$.) Points in the remainder of region III (and those points in region I which lie within the fast branch of the normal speed locus) will have their single-napped cones lying both upstream and downstream. If, and only if, the normal speed loci and characteristic loci coincide (as for the wave equation) will all equilibria have

their forward-facing cones downstream.

The proof that Q (equation (13)) has two real roots is similar to the proof given earlier. Equation (14) gives the boundary of region III-A as the set of all points for which the highest coefficient of Q vanishes. We then see by checking a point inside region III-A, such as a perpendicular flow, that for points in region III-A, Q has its leading coefficient negative. Q is nonnegative for $\lambda = 0$, where it equals $\frac{u_0^4 \sin^2\phi \sin^2\theta}{m^2 M^2}$, so Q has at least two real roots. This implies, by the same argument as above, that the only real double point must be at $\sin\theta = 0$, if Q has two nonreal roots for all θ. The characteristic plane corresponding to this double point is $x = 0$. This is the plane of $\vec{u}_0$ and $\vec{A}_0$, as we found in the other coordinate system.

The polynomial has two nonreal roots for perpendicular flow, i.e. $\cos\phi = 0$. In this case, the polynomial is second order in λ^2 and, by exactly the same argument we used for perpendicular flow in the other coordinate system, we find two real and two nonreal roots. The result is then true in a neighborhood of $\cos\phi = 0$, by the previous argument.

We now show, following Rellich [10], that the roots of the characteristic polynomial are analytic in θ for $0 \leq \theta < 2\pi$ if the polynomial has two real and two nonreal roots for all θ and only a finite number of real double points. The only difficulty is at a point θ_0 where the real roots coincide, since the roots of a polynomial with

analytic coefficients, noncoincident roots, and leading coefficient nonzero are analytic. Let the real roots be $\lambda_{1,2}(\theta)$ with $\lambda_1(\theta_0) = \lambda_2(\theta_0)$. The functions $\lambda_{1,2}(\theta)$ have expansions in $(\theta-\theta_0)^{1/2}$ about θ_0 since the complex roots are bounded away from the real axis. The terms $(\theta-\theta_0)^{n/2}$, n odd, do not appear in these expansions, since $\lambda_{1,2}(\theta)$ are real for both $\theta-\theta_0 \geq 0$ and $\theta-\theta_0 \leq 0$. For if $\lambda_1(\theta) = \sum_{n=0}^{\infty} a_{n/2}(\theta-\theta_0)^{n/2}$, and $a_{m/2}$, m odd, is the first nonzero coefficient of the form $a_{n/2}$, n odd, then

$$\lambda_1(\theta) - \sum_{n=0}^{m-1} a_{n/2}(\theta - \theta_0)^{n/2}$$

is a real function for θ real. However, its imaginary part, to leading order for $(\theta-\theta_0)$ small, will be $\operatorname{Im} a_{m/2}(\theta-\theta_0)^{m/2}$, which will be nonzero for either $\theta-\theta_0 > 0$ or $\theta-\theta_0 < 0$, unless $a_{m/2} = 0$, proving the result.

We close this section with a useful simple property of the characteristic polynomial in any Cartesian coordinate system. By equation (10) and the explicit form of Q, equation 9, we find

$$
\begin{aligned}
&Q_\lambda(\cos(\theta+\pi), \sin(\theta+\pi), \lambda_i(\theta+\pi)) \\
&\qquad = Q_\lambda(-\cos\theta, -\sin\theta, -\overline{\lambda_i(\theta)}) \\
&\qquad = -\overline{Q_\lambda(\cos\theta, \sin\theta, \lambda_i(\theta))}\ , \qquad i = 1,2,3,4.
\end{aligned} \tag{15}
$$

Section 5. The Derivation of the Solution

A. Statement of the Problem

In this section, we find a solution to equation (7) which satisfies the causality condition and the smoothness and growth at infinity conditions given below. The steady flow past a source is then given by forming the proper superpositions of this solution, using various values of α, β, and γ, and proceeding as shown earlier. We begin by describing the differential equation and the conditions we impose on its solution.

The differential equation has the form

$$(16) \quad Q(\partial/\partial x, \partial/\partial y, \partial/\partial z)u(x,y,z) = \delta^{\alpha}(x)\ \delta^{\beta}(y)\ \delta^{\gamma}(z) ,$$

$$\alpha + \beta + \gamma = 6 \text{ or } 7,\ \alpha,\beta,\gamma \geq 1 ,$$

where Q is a homogeneous polynomial of fourth degree with constant coefficients which depend upon the Cartesian coordinate system chosen. Such properties of equation (16) as the number of roots $\lambda(\theta)$ of its characteristic polynomial $Q(\cos\theta, \sin\theta, \lambda)$ (i.e. whether or not $Q(0,0,1)$ vanishes) and the number of real roots of this polynomial are not invariant under arbitrary rotations of the coordinate system. We have shown that in a large part of region III (and possibly in all of region III), we may choose at least one coordinate system for each point, i.e. for each equilibrium, such that

equation (16) has the following properties: $a_0 = Q(0,0,1) \neq 0$, and for every unit vector $\vec{n} = (\cos\theta, \sin\theta)$, the characteristic polynomial $Q(\cos\theta, \sin\theta, \lambda)$ $(\equiv Q(\vec{n},\lambda))$ has roots $-\lambda_i(\vec{n})$ $(\equiv -\lambda_i(\theta))$, $i = 1,2,3,4$ with $\lambda_{1,2}(\vec{n})$ real and $\lambda_{3,4}(\vec{n})$ complex, and $\mathrm{Im}\,\lambda_3 > 0$. The roots all satisfy $\lambda_i(\theta+\pi) = -\overline{\lambda_i(\theta)}$ and $Q_\lambda(\vec{n}(\theta+\pi), -\lambda_i(\theta+\pi)) = -\overline{Q_\lambda(\vec{n}(\theta), -\lambda_i(\theta))}$. The roots are moreover distinct except at two double points, $\theta = \theta_0$ and $\theta = \theta_0 + \pi$, $0 \leq \theta < 2\pi$, where the real roots coincide. In addition, the coordinate system can be chosen so as to place the single-napped forward-facing characteristic cone given by the construction of Section 3-E in the space $z > 0$. The causality condition then takes the form that the solution be smooth for $z < 0$. For definiteness, we will assume that the coordinate system chosen is one of the two explicitly given in Section 4, which places the double point at $\sin\theta = 0$ where $\lambda_{1,2} = 0$. We note that the arguments we shall use may be extended simply to the case of a finite number of double points where the real roots coincide.

We require the solution of equation (16) to be smooth except for possible singularities across the characteristic cone in the space $z > 0$. We expect that there will be a solution to equation (16) which is homogeneous of degree -2 (-3) for $\alpha+\beta+\gamma = 6$ (7) since $\delta^\alpha(x)\delta^\beta(y)\delta^\gamma(z)$ is homogeneous of degree $-(\alpha+\beta+\gamma)$ and $Q(\partial/\partial x, \partial/\partial y, \partial/\partial z)$ is a homogeneous operator of fourth degree. We therefore require

the solution to have this rate of decay at infinity except, of course, along the characteristic cone in $z > 0$.

We will not prove that the solution we obtain is unique under the above conditions although we believe it is; one difficulty is the dependence of the form of the equation on the coordinate system chosen.

We note that the complex roots $-\lambda_{3,4}(\vec{n})$ of the characteristic polynomial occur because the equilibrium velocity $\vec{u}_0$ is subsonic with respect to the fast disturbance speed of the time dependent problem. We therefore do not expect the solution of equation (16) to vanish for $z < 0$, as would occur for equilibria in region I. We will, however, not need this as an additional condition on our solution.

It will be convenient for us to solve equation (16) with smaller exponents on the delta functions and then differentiate appropriately to find the solution. We shall use both $\alpha + \beta + \gamma = 5$, $\alpha, \beta, \gamma \geq 1$, and $\alpha = \beta = \gamma = 1$.

B. The Method of Fourier Transforms

The use of Fourier transforms for the solution of the above problem will be found wanting in several respects. Difficulties arise both in interpreting the result, which is written in terms of divergent integrals, and in showing the solution is indeed correct i.e. justifying such procedures as the displacement of the contour, etc. The interpretation

of the divergent integrals is a serious difficulty in that the correct interpretation should give a surface of singularity, namely the surface formed by the tangent half lines to the line joining the slow loci in the construction of the characteristic cone, and this surface does not appear in the usual treatment of the above divergent integrals, i.e. the finite part integral considered by various authors, e.g. Hadamard [11]. In fact, this last method is normally expected to represent an ordinary function, while we expect a solution which is homogeneous of degree -2 or -3, so it will not be regular at the origin and must be expected to be a distribution. The method we shall use later will give a solution in terms of derivatives of continuous functions so it is a well defined distribution. In addition, our method will lead in a natural way to a proof that the solution is indeed correct. However, Fourier transforms have been so often useful in the past that it will be interesting to use them here for comparison with our later techniques.

We begin by seeking a solution $v(x,y,z)$ to equation (16) with $\alpha+\beta+\gamma = 5$, $\alpha,\beta,\gamma \geq 1$. We transform this equation in the usual way, assuming that $v(x,y,z)$ is such that the boundary terms at infinity all vanish (even though we expect that the derivatives of v will be singular across the characteristic cone in $z > 0$). We solve for $v(k_x,k_y,\omega)$, the Fourier transform of $v(x,y,z)$, and the proposed solution is then given by the inverse transform, namely:

$$v(x,y,z) = \mathrm{Re}\ \frac{-1}{(2\pi)^3} \iiint_{-\infty}^{+\infty} dk_x\ dk_y\ d\omega\ e^{i(\vec{k}\cdot\vec{x}+\omega z)}\ \frac{k_x^{\alpha-1} k_y^{\beta-1} \omega^{\gamma-1}}{Q(k_x,k_y,\omega)}$$

with $\vec{k} = (k_x,k_y) = k\vec{n}$, $\vec{n} = (\cos\theta, \sin\theta)$, and $\vec{x} = (x,y)$. This integral is singular because $Q(\vec{k},\omega)$ has the real roots $\omega_{1,2}(\vec{k}) = -k\ \lambda_{1,2}(\vec{n})$. For each $k \neq 0$, we displace the ω contour from the real axis to a contour c parallel to the real axis in the lower half plane and above the pole at $\omega = \omega_3(\vec{k}) = -k\ \lambda_3(\vec{n})$. This is the proper displacement if $v(x,y,z)$ is to satisfy the causality condition, as will be shown later. The methods we use to evaluate the resulting integral will follow closely those used by H. Weitzner for the time dependent problem [8].

The integral over ω may be evaluated by residues to give for $z > 0$,

$$v(x,y,z) = \mathrm{Re}\ \frac{-i(-1)^{\gamma-1}}{(2\pi)^2} \iint_{-\infty}^{+\infty} dk_x\ dk_y \sum_{j=1,2,4} \cos^{\alpha-1}\theta\ \sin^{\beta-1}\theta$$

$$\cdot \frac{\lambda_j^{\gamma-1}(\vec{n})\ e^{ik(\vec{x}\cdot\vec{n}-\lambda_j(\vec{n})z)}}{k\ Q_\lambda(\vec{n},-\lambda_j(\vec{n}))}$$

and for $z < 0$,

$$v(x,y,z) = \mathrm{Re}\ \frac{+i(-1)^{\gamma-1}}{(2\pi)^2} \iint_{-\infty}^{+\infty} dk_x\ dk_y\ \cos^{\alpha-1}\theta\ \sin^{\beta-1}\theta$$

$$\cdot \frac{\lambda_3^{\gamma-1}(\vec{n})\ e^{ik(\vec{x}\cdot\vec{n}-\lambda_3(\vec{n})z)}}{k\ Q_\lambda(\vec{n},-\lambda_3(\vec{n}))}$$

where we have used the homogeniety of Q to give $\omega_j(\vec{k}) = -k\,\lambda_j(\vec{n})$ and $Q_\lambda(\vec{k},-\omega) = k^3 Q_\lambda(\vec{n},-\omega/k)$.

Considering first $z > 0$, we change the variables of integration to polar coordinates (k,θ), $-\pi \leq \theta < \pi$, and then let $k' = -k$, $\theta' = \theta + \pi$ on $(-\pi,0)$ to find

$$v(x,y,z) = \mathrm{Re}\,\frac{-i(-1)^{\gamma-1}}{(2\pi)^2}\int_0^\pi d\theta \int_{-\infty}^{+\infty} dk\ \mathrm{sgn}\,(k)$$

$$\cdot \sum_{j=1,2,4} \frac{\cos^{\alpha-1}\theta\ \sin^{\beta-1}\theta\ \lambda_j^{\gamma-1}(\vec{n})\ e^{ik(\vec{x}\cdot\vec{n}-\lambda_j(\vec{n})z)}}{Q_\lambda(\vec{n},-\lambda_j(\vec{n}))},$$

$$z > 0.$$

The integral over k exists as a distribution and may be evaluated using

$$\int_{-\infty}^{\infty} dk\ \mathrm{sgn}\,(k)\ e^{ikt} = \frac{2i}{t}\ ,$$

[12], to give

$$v(x,y,z) = \mathrm{Re}\,\frac{\pm 2(-1)^{\gamma-1}}{(2\pi)^2}\sum_{j=1,2,4}\int_0^\pi d\theta\ \frac{\cos^{\alpha-1}\theta\ \sin^{\beta-1}\theta\ \lambda_j^{\gamma-1}(\vec{n})}{(\vec{x}\cdot\vec{n}-\lambda_j(\vec{n})z)Q_\lambda(\vec{n},-\lambda_j(\vec{n}))},$$

$$z > 0.$$

and, in the same way,

$$v(x,y,z) = \mathrm{Re}\, \frac{-2(-1)^{\gamma-1}}{(2\pi)^2} \int_0^{\pi} d\theta\, \frac{\cos^{\alpha-1}\theta\, \sin^{\beta-1}\theta\, \lambda_3^{\gamma-1}(\vec{n})}{(\vec{x}\cdot\vec{n}-\lambda_3(\vec{n})z)Q_\lambda(\vec{n},-\lambda_3(\vec{n}))},$$

$$z < 0 .$$

We may rewrite the above integrals as integrals over the unit circle by using the relations $\lambda_i(\theta-\pi) = -\overline{\lambda_i(\theta)}$, $Q_\lambda(\overline{n(\theta-\pi)},-\lambda_i(\theta-\pi)) = -Q_\lambda(\overline{n(\theta)},-\lambda_i(\theta))$, $i=1,2,3,4$ (equations (10) and (15)) and $\vec{x}\cdot\vec{n}\big|_{\theta-\pi} = -\vec{x}\cdot\vec{n}\big|_{\theta}$, which give

(17)

$$v(x,y,z) = \mathrm{Re}\, \frac{(-1)^{\gamma-1}}{(2\pi)^2} \int_0^{2\pi} d\theta \sum_{j=1,2,4} \frac{\cos^{\alpha-1}\theta\, \sin^{\beta-1}\theta\, \lambda_j^{\gamma-1}(\vec{n})}{(\vec{x}\cdot\vec{n}-\lambda_j(\vec{n})z)Q_\lambda(\vec{n},-\lambda_j(\vec{n}))},$$

$$z > 0,$$

$$v(x,y,z) = \mathrm{Re}\, \frac{(-1)^{\gamma}}{(2\pi)^2} \int_0^{2\pi} d\theta\, \frac{\cos^{\alpha-1}\theta\, \sin^{\beta-1}\theta\, \lambda_3^{\gamma-1}(\vec{n})}{(\vec{x}\cdot\vec{n}-\lambda_3(\vec{n})z)Q_\lambda(\vec{n},-\lambda_3(\vec{n}))},$$

$$z < 0 .$$

The conjugation signs do not appear as we take the real part.

We have the proposed solution expressed in equation (17) in terms of divergent integrals. We shall, by using a different method of solution, find the proper interpretation of these integrals to be that given by formally integrating with respect to x, y, and z, as follows: We introduce the notation

$$(18)\quad P(m,j) = \int_0^{2\pi} d\theta\, \frac{(\vec{x}\cdot\vec{n}-\lambda_j(\vec{n})z)^m}{Q_\lambda(\vec{n},-\lambda_j(\vec{n}))} \log G_j(\vec{x}\cdot\vec{n} - \lambda_j(\vec{n})z)$$

where m is a positive integer, j = 1,2,3,4, and

$$G_j(x) = \begin{cases} |x| & \text{for } j = 1,2 \\ x & \text{for } j = 3,4 \end{cases} .$$

The solution is then

$$v(x,y,z) = \operatorname{Re} \left(\frac{\partial}{\partial x}\right)^{\alpha-1} \left(\frac{\partial}{\partial y}\right)^{\beta-1} \left(\frac{\partial}{\partial z}\right)^{\gamma-1} \left[\frac{1}{(2\pi)^2} \sum_{j=1,2,4} P(1,j)\right], \qquad z > 0$$

(19)

$$v(x,y,z) = -\operatorname{Re}\left(\frac{\partial}{\partial x}\right)^{\alpha-1} \left(\frac{\partial}{\partial y}\right)^{\beta-1} \left(\frac{\partial}{\partial z}\right)^{\gamma-1} \left[\frac{1}{(2\pi)^2} P(1,3)\right], \qquad z < 0.$$

We shall discuss this solution at length later, giving the branches for the logarithms. In particular, we shall show that the functions $P(1,1) + P(1,2)$, $P(1,3)$, and $P(1,4)$ are continuous in x, y, and z, and thus $v(x,y,z)$ is a well defined distribution. The solution $u(x,y,z)$ of equation (16) is then found by taking the appropriate derivatives of $v(x,y,z)$.

We conclude this discussion of the Fourier transform method by observing that if we had displaced the ω contour in the above derivation into the upper half plane, the proposed solution v_u would be

$$v_u = \operatorname{Re} \frac{1}{(2\pi)^2} \left(\frac{\partial}{\partial x}\right)^{\alpha-1} \left(\frac{\partial}{\partial y}\right)^{\beta-1} \left(\frac{\partial}{\partial z}\right)^{\gamma-1} P(1,4), \qquad z > 0$$

and

$$v_u = -\operatorname{Re} \frac{1}{(2\pi)^2} \left(\frac{\partial}{\partial x}\right)^{\alpha-1} \left(\frac{\partial}{\partial y}\right)^{\beta-1} \left(\frac{\partial}{\partial z}\right)^{\gamma-1} \sum_{j=1,2,3} P(1,j), \qquad z < 0.$$

The inclusion of the sum over the real roots for

$z < 0$ will cause singularities to appear across the characteristic cone in that half space, as will be shown later. This violates the causality condition and this otherwise legitimate solution must be rejected.

C. The Method of Solution and an Example

We shall find a solution to equation (16) by extending the plane wave representation for the fundamental solution of homogeneous hyperbolic equations, as given by F. John [13], to a representation for the solution of equation (16) with $\alpha = \beta = \gamma = 1$.

We begin by assuming for the moment that equation (16) is hyperbolic i.e. that all the roots $\lambda_i(\vec{n})$ of $Q(\vec{n},\lambda)$ are real. The solution to equation (16) with $\alpha = \beta = \gamma = 1$, using Duhamel's principle, is $H(z)u_1(x,y,z)$, where $H(z)$ is the Heaviside function and $u_1(x,y,z)$ is a solution of the following Cauchy problem in the space $z \geq 0$:

$$Q\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)u_1(x,y,z) = 0 , \qquad z > 0$$

(20) with initial conditions

$$\frac{a_0 \partial^i u_1(x,y,0^+)}{\partial z^i} = \begin{cases} 0 & \text{for } i = 0,1,2 \\ \delta(x)\delta(y) & \text{for } i = 3 \end{cases}$$

The solution u_1 of this problem is given by F. John [13] in terms of the two dimensional Laplacian ($\Delta_{\vec{x}} = D_x^2 + D_y^2$) iterated three times acting upon a superposition of plane wave functions, i.e. functions

of the form $g(\vec{x}\cdot\vec{n}-\lambda_i(\vec{n})z)$. In particular, for the case of two space dimensions (such as here with z regarded as the time variable), the plane wave functions are $(\vec{x}\cdot\vec{n}-\lambda_i(\vec{n})z)^m \log|\vec{x}\cdot\vec{n}-\lambda_i(\vec{n})z|$, where m is a positive integer. For $\lambda_i(\vec{n})$ complex, these functions are no longer plane wave functions [since $|\vec{x}\cdot\vec{n}-\lambda_i(\vec{n})z|^2 = (\vec{x}\cdot\vec{n}-\lambda_i(\vec{n})z)(\overline{\vec{x}\cdot\vec{n}-\lambda_i(\vec{n})z})$] and u_1 is no longer a solution to the homogeneous equation in $z > 0$. We correct this by replacing $\log|\vec{x}\cdot\vec{n}-\lambda_i(\vec{n})z|$ by the plane wave functions $\log(\vec{x}\cdot\vec{n}-\lambda_i(\vec{n})z)$ for $i = 3,4$, where we use the fact that the roots $-\lambda_i(\vec{n})$ of the polynomial $Q(\vec{n},\lambda)$ are analytic functions of θ to extend the logarithms into the complex plane. We will, of course, have to give branches for the logarithms. We make this change in $u_1(x,y,z)$, take the real part, and arrive at a new function u_2, where $Q(\partial/\partial x, \partial/\partial y, \partial/\partial z)u_2 = 0$ in $z > 0$ (with $\lambda_{3,4}(\vec{n})$ complex).

The function u_2 unfortunately does not satisfy the initial conditions of equation (20). In fact, we will have

$$\frac{a_0\,\partial^i u_2(x,y,0^+)}{\partial z^i} = \begin{cases} f_i(x,y) & \text{for } i=0,1,2 \\ f_3(x,y)+\delta(x)\delta(y) & \text{for } i=3, \end{cases}$$

with $f_i \not\equiv 0$, $i = 1,2,3,4$. The function $H(z)u_2$ and its first three normal derivatives will then not be continuous across $z = 0$, $x^2 + y^2 \neq 0$, and $H(z)u_2$ will not satisfy equation (16). We correct this by finding a function $M(x,y,z)$ which is a solution to the homogeneous equation

for $z \neq 0$ and which will smooth $H(z)u_2$, i.e. so that $u_3 = H(z)u_2 + M$ will have at least three continuous normal derivatives across $z = 0$, $x^2 + y^2 \neq 0$. The function M may be found by assuming it to be a superposition of plane wave functions similar in form to u_2, but using only the plane waves $\vec{x} \cdot \vec{n} - \lambda_{3,4}(\vec{n})z$ in order not to violate the causality condition. The arbitrary functions included in this superposition will be determined so that u_2 and its first three normal derivatives are continuous across $z = 0$, $x^2 + y^2 \neq 0$. The function u_3 will then have the property that its jumps, $\frac{\partial^n}{\partial z^n} u_3(x,y,0^+) - \frac{\partial^n u_3}{\partial z^n}(x,y,0^-)$, across $z = 0$ vanish for $n = 0,1,2$, and equal $\frac{1}{a_0}\delta(x)\delta(y)$ for $n = 3$. We can then easily prove that u_3 is a solution to equation (16) which satisfies the smoothness, causality, and rate of decay at infinity conditions given above.

We illustrate this method by using it to find the fundamental solution to Laplace's equation in three dimensions, i.e. a function $u(x,y,z)$ such that $u_{xx} + u_{yy} + u_{zz} = \delta(x)\delta(y)\delta(z)$. The characteristic polynomial is $Q(\vec{n},\lambda) = \lambda^2 + 1$, with roots $-\lambda_{\pm}(\vec{n}) = \pm i$. If the roots were real, the fundamental solution would be $H(z)u_1$ where, with $\Delta_{\vec{x}} = D_x^2 + D_y^2$,

$$H(z)u_1 = H(z)\frac{\Delta^2_{\vec{x}}}{3!(2\pi)^2}\int_0^{2\pi} d\theta \left[\frac{(\vec{x}\cdot\vec{n}-\lambda_+ z)^3 \log|\vec{x}\cdot\vec{n}-\lambda_+ z|}{2\lambda_+} + \frac{(\vec{x}\cdot\vec{n}-\lambda_- z)^3 \log|\vec{x}\cdot\vec{n}-\lambda_- z|}{2\lambda_-} \right].$$

With $-\lambda_{\pm} = \pm\, i$, $H(z)u_1$ is not a solution of the homogeneous equation for $z > 0$, so we extend the logarithms into the complex plane to find the new function

$$H(z)u_2 = \text{Re}\, \frac{H(z)\, \Delta^2_{\vec{x}}}{3!(2\pi)^2 2i} \int_0^{2\pi} d\theta\, [(\vec{x}\cdot\vec{n} + iz)^3 \log(\vec{x}\cdot\vec{n} + iz)$$

$$- (\vec{x}\cdot\vec{n} - iz)^3 \log(\vec{x}\cdot\vec{n} - iz)]\,.$$

The logarithms are defined to be the principal branches cut on the negative real axis i.e. $-\pi < \text{Im}\log z \leq \pi$. This function is not smooth across the plane $z = 0$, $x^2 + y^2 \neq 0$. We then seek a function M which is a homogeneous solution for $z \neq 0$ and such that $u_3 = H(z)u_2 + M$ and its first three normal derivatives are continuous across $z = 0$, $x^2 + y^2 = 0$. We assume M has the form

$$M = \text{Re}\, \Delta^2_{\vec{x}} \int_0^{2\pi} d\theta\, [a(\theta)(\vec{x}\cdot\vec{n} + iz)^3 \log(\vec{x}\cdot\vec{n} + iz)$$

$$+ b(\theta)(\vec{x}\cdot\vec{n} - iz)^3 \log(\vec{x}\cdot\vec{n} - iz)]$$

with the logarithms defined as above. This function is a solution of the homogeneous equation for $z \neq 0$ since it is a superposition of plane waves. We now pick $a(\theta)$ and $b(\theta)$ so that $H(z)u_2 + M$ and its first three normal derivatives are continuous across $z = 0$. A method for finding such functions will be given when we consider equation (16) and we don't wish to enter the details in this example. We will thus arrive at the function

$$M = -\operatorname{Re} \frac{\Delta_{\vec{x}}^2}{2(2\pi)^2 3!\,2i} \int_0^{2\pi} d\theta \, [(\vec{x}\cdot\vec{n}+zi)^3 \log(\vec{x}\cdot\vec{n}+zi) - (\vec{x}\cdot\vec{n}-zi)^3 \log(\vec{x}\cdot\vec{n}-zi)] \,.$$

The proposed solution is $u_3 = H(z)u_2 + M$ which we may evaluate by applying the Laplacians under the integral sign for $z \neq 0$. We then use the identity,

$$\int_0^{2\pi} \frac{d\theta}{\cos\theta + bi} = -\frac{2\pi i \operatorname{sgn}(b)}{\sqrt{1+b^2}} \,,$$

which is true for b real and nonzero, to find

$$u_3 = -\frac{1}{4\pi\sqrt{x^2+y^2+z^2}} \,.$$

This, as is well known, is the correct answer. In the case of equation (16), we shall have to prove that the solution we obtain using this method is correct.

D. Derivation of the Solution

We apply the above method to equation (16) with $\alpha = \beta = \gamma = 1$. To begin, the solution $H(z)u_1$ to the hyperbolic problem is modified by extending the logarithms involving the complex roots $\lambda_{3,4}(\theta)$ into the complex plane. We then show that the resulting function $H(z)u_2$ is well defined, using essentially the same method employed by F. John [13] to show that $H(z)u_1$ is well defined. We find the "smoothing" function M, and $u_3 = H(z)u_2 + M$ will be the proposed solution. The proposed solution u_3 is shown to be the same function as that found formally by the Fourier

transform method (equation (19)). Finally, we check that we have not neglected another solution during one of the steps in the derivation of M. (This is not a uniqueness proof.)

We begin with the function $H(z)u_1$ which would be the solution if all the roots $\lambda_i(\vec{n})$, $i = 1,2,3,4$ were real. We take this function, as given by F. John [13], extend the logarithms $\log |\vec{x}\cdot\vec{n} - \lambda_{3,4}(\vec{n})z|$ into the complex plane, and take the real part of the resulting expression to find

$$(21) \qquad H(z)u_2(x,y,z) = \frac{H(z)}{(2\pi)^2 7!} \operatorname{Re} \Delta^3_{\vec{x}} \sum_{j=1}^{4} P(7,j)$$

where $P(m,j)$ was introduced above (equation (18)). Branches for the logarithms in $P(7,3)$ and $P(7,4)$ are given by setting $\log 1 = 0$, and placing the branch cut for $\log(\vec{x}\cdot\vec{n} - \lambda_3(\vec{n})z)$ in the closed upper half plane and for $\log(\vec{x}\cdot\vec{n} - \lambda_4(\vec{n})z)$ in the closed lower half plane. The logarithms will then be continuous for $z > 0$ since $\operatorname{Im} \lambda_3(\vec{n}) = - \operatorname{Im} \lambda_4(\vec{n}) > 0$. For definiteness, we take both branch cuts along the negative real axis.

We next show that $u_2(x,y,z)$ is well defined despite the double points of the real roots at $\sin\theta = 0$ at which points the denominators of $P(7,1)$ and $P(7,2)$ vanish. Each of these integrals is singular; however, their sum $P(7,1) + P(7,2)$ or, more precisely, S, the sum of their integrands, is continuous and we shall always use these

integrals in this combination.

The continuity (and differentiability for $m > 2$) of the sum may be shown, following F. John [13], more generally for $P(m,1) + P(m,2)$ with $m \geq 2$. We first note that, for $\sin\theta \neq 0$, a residue expansion gives

$$S = \sum_{i=1,2} \frac{(\vec{x}\cdot\vec{n}-\lambda_i(\vec{n})z)^m \log(\vec{x}\cdot\vec{n}-\lambda_i(\vec{n})z)}{Q_\lambda(\vec{n},-\lambda_i(\vec{n}))}$$

$$(22) \qquad = -\sum_{i=3,4} \frac{(\vec{x}\cdot\vec{n}-\lambda_i(\vec{n})z)^m \log(\vec{x}\cdot\vec{n}-\lambda_i(\vec{n})z)}{Q_\lambda(\vec{n},-\lambda_i(\vec{n}))}$$

$$+ \frac{1}{2\pi i}\int_c d\lambda \, \frac{(\vec{x}\cdot\vec{n}-\lambda z)^m \log(\vec{x}\cdot\vec{n}-\lambda z)}{Q(\vec{n},\lambda)}$$

The path c is a circle of radius R about the point $\lambda_0 = \frac{\vec{x}\cdot\vec{n}}{z}$ slit along the parallel to the positive imaginary axis through λ_0 for $\mathrm{Re}\,\lambda_3 \neq \lambda_0$ and along some other radius, say $\arg(\lambda-\lambda_0) = \frac{\pi}{4}$, for $\mathrm{Re}\,\lambda_3 = \lambda_0$, or c may be any equivalent contour. R should be so large that c contains all roots of $Q(\vec{n},\lambda)$, e.g.

$$R = |\lambda_0| + 2\max_i |\lambda_i(\vec{n})| , \qquad i = 1,2,3,4.$$

Along the slit $(\vec{x}\cdot\vec{n} - \lambda z)^m = (\lambda - \lambda_0)^m z^m$ while $Q(\vec{n},\lambda)$ vanishes at most like $(\lambda - \lambda_0)^2$, when the real roots coincide at λ_0, i.e. for $\lambda_0 = 0$. The integral in equation (6) is thus continuous in x,y, and z, since $m \geq 2$. The sum over the complex roots on the right side of equation (22) is infinitely differentiable in x,y, and z, since the

denominators are both nonzero and $\lambda_{3,4}(\vec{n})$ are nonreal. The function S is thus continuous. In order to form derivatives with m > 2, we write the integral in equation (22) as

$$\frac{z^m}{2\pi i} \int_{|\omega|=R} d\omega \, \frac{\omega^m \log(\omega z)}{Q(\vec{n},\omega+\lambda_0)} - z^m \int_{\lambda_0}^{\lambda_0+iR} \frac{d\omega \, \omega^m}{Q(\vec{n},\omega+\lambda_0)}$$

for Re $\lambda_3 \neq \lambda_0$, and integrate the second integral above over arg $(\lambda - \lambda_0) = \frac{\pi}{4}$ for Re $\lambda_3 = \lambda_0$. The partial derivatives with respect to x, y, and z of these integrals (recalling that $\lambda_0 = \frac{\vec{x} \cdot \vec{n}}{z}$) may be computed by differentiating under the integral signs. If we then integrate by parts, the resulting expressions will not contain any contributions from the limits of integration.

We thus see that the integral in equation (22), and hence S and P(m,1) + P(m,2), has in the space z > 0 all derivatives with respect to x, y, and z of order $\leq$ m-2 and these derivatives are continuous.

We can moreover show that these derivatives have finite limits as $z \to 0^+$, for $x^2 + y^2 \neq 0$. This may be seen by noting that as $z \to 0^+$, we have $\lambda_0 \to \infty$. Therefore, for z_0 sufficiently small, we may take our path of integration for the integral in equation (22) to be a circle about all the roots of Q where this circle does not contain λ_0. The derivatives and the limit $z \to 0^+$ may then be applied directly under the integral sign.

In the remainder of this thesis, the integrand of $P(m,1) + P(m,2)$, $m \geq 2$, is to be regarded as defined as in equation (22).

The function $H(z)u_2$, equation (21), for $z > 0$ is given in terms of the Laplacian three times iterated applied to a five time continuously differentiable function. The terms $P(7,1) + P(7,2)$ are five times continuously differentiable by the argument above and there is no difficulty differentiating $P(7,3)$ and $P(7,4)$. The function $H(z)u_2$ is then a well defined distribution and since it is a superposition of plane wave functions, it satisfies the homogeneous equation for $z > 0$. We are using the fact that $H(z)u_2$ is a distribution to interchange the operator Q with the Laplacians.

We next seek a function M such that M satisfies the homogeneous equation for $z \neq 0$ and such that $u_3 = H(z)u_2+M$ and its first three normal derivatives are continuous across $z = 0$, $x^2+ y^2 \neq 0$. We assume M is of the form

$$M(x,y,z) = \frac{1}{7!(2\pi)^2} \operatorname{Re} \Delta^3_{\vec{x}} \left[\int_0^{2\pi} d\theta \right.$$

$$\cdot\, a(\theta)(\vec{x}\cdot\vec{n}-\lambda_3(\vec{n})z)^7 \log (\vec{x}\cdot\vec{n}-\lambda_3(\vec{n})z)$$

$$\left. +b(\theta)(\vec{x}\cdot\vec{n}-\lambda_4(\vec{n})z)^7 \log (\vec{x}\cdot\vec{n}-\lambda_4(\vec{n})z) \right]$$

for $z \neq 0$, where $a(\theta)$ and $b(\theta)$ are to be determined.

We assume, without loss of generality, that $a(\theta)$ and $b(\theta)$ are continued in θ with period 2π, i.e. $a(\theta+2\pi) = a(\theta)$ and $b(\theta+2\pi) = b(\theta)$. We do not include integrals over the real roots, $\lambda_{1,2}(\vec{n})$, in M as this will introduce singularities across the characteristic cone in $z < 0$, violating the causality condition. We take the same branches for the logarithms here as were taken for the logarithms in $H(z)u_2$, equation (21). This choice is made to avoid introducing singularities in M for $z \neq 0$ due to branch cuts and so that all logarithms we consider will be defined in the same way.

The functions $a(\theta)$ and $b(\theta)$ will now be determined by the condition that $H(z)u_2 + M$ and its first three z derivatives be continuous across $z = 0$, $x^2 + y^2 \neq 0$. (We will show later that $H(z)u_2 + M$ is analytic in a neighborhood of $z = 0$, $x^2 + y^2 \neq 0$.) We begin by finding the limits of u_3 and its first three z derivatives as $z \to 0^{\pm}$, $x^2 + y^2 \neq 0$. The various derivatives and limits of u_3 may be taken under the Laplacians since u_3 is a distribution. They may then be taken under the integral signs, since we have shown the integrals appearing in u_2 are five times differentiable for $z > 0$ and these derivatives are moreover continuous as $z \to 0^+$, $x^2 + y^2 \neq 0$. The integrals appearing in M may be treated in the same way as they present no difficulties.

The limits as $z \to 0^+$ of $H(z)u_2$ and its first three

z derivatives are given by

$$\frac{\partial^n H(z)u_2(x,y,0^+)}{\partial z^n} = \frac{1}{(2\pi)^2}\,\mathrm{Re}\,\Delta^3_{\vec{x}}\,\frac{1}{(7-n)!}\int_0^{2\pi} d\theta\,(\vec{x}\cdot\vec{n})^{7-n}$$

$$\cdot\,\log|\vec{x}\cdot\vec{n}|\sum_{i=1}^{4}\frac{(-\lambda_i(\theta))^n}{Q_\lambda(\theta,-\lambda_i(\theta))}$$

(23)

$$+\,\mathrm{Re}\,\Delta^3_{\vec{x}}\,\frac{i}{4\pi(7-n)!}\int_{\substack{\vec{x}\cdot\vec{n}\leq 0\\ 0\leq\theta\leq 2\pi}} d\theta\,(\vec{x}\cdot\vec{n})^{7-n}\left(-\frac{(-\lambda_3(\theta))^n}{Q_\lambda(\theta,-\lambda_3(\theta))}+\frac{(-\lambda_4(\theta))^n}{Q_\lambda(\theta,-\lambda_4(\theta))}\right)$$

$$+\,\mathrm{Re}\,\Delta^3_{\vec{x}}\,\frac{c_n}{(2\pi)^2 7!}\int_0^{2\pi} d\theta\,(\vec{x}\cdot\vec{n})^{7-n}\sum_{i=1}^{4}\frac{(-\lambda_i(\theta))^n}{Q_\lambda(\theta,-\lambda_i(\theta))}\quad,n=0,1,2,3$$

with $c_0 = 0$, $c_1 = 1$, $c_2 = 13$, and $c_3 = 55$.

Before we proceed in a similar way with the function M, we will simplify the above expression. We use the following identity:

$$(24)\quad \frac{1}{2\pi i}\int_c \frac{\lambda^q\,d\lambda}{Q(\lambda)} = \sum_{i=1}^{m}\frac{\lambda_i^q}{Q_\lambda(\lambda_i)} = \begin{cases} 0 & \text{for } q = 0,1,..m-2 \\ \dfrac{1}{a_0} & \text{for } q = m-1 \end{cases}$$

where $Q(\lambda)$ is a polynomial of degree m with leading coefficient $a_0 \neq 0$ and with all roots λ_i simple. The path of integration c is any contour enclosing all the roots of Q. For Q with coincident roots, as in our case, we define the summation in equation (24) by the integral and the identity still holds. Applying identity (24) to

equation (23) with $n = 0,1,2$, we see that the first and third terms on the right side vanish. With $n=3$, and using identity (24), we find these two terms are the following:

$$\frac{\mathrm{Re}}{a_0 4!(2\pi)^2}\,\Delta^3_{\vec{x}}\int_0^{2\pi} d\theta\,[(\vec{x}\cdot\vec{n})^4 \log|\vec{x}\cdot\vec{n}| + \frac{c_3 4!}{7!}(\vec{x}\cdot\vec{n})^4]$$

$$(25) = \frac{1}{a_0 4!(2\pi)^2}\,\Delta^3_{\vec{x}}\; r^4 \log r \int_0^{2\pi} d\theta \cos^4\theta =$$

$$= \frac{1}{2\pi a_0}\,\Delta_{\vec{x}} \log r = \frac{\delta(x)\delta(y)}{a_0}$$

We note here that if all the roots $\lambda_i(\theta)$, $i=1,2,3,4$, were real, the two terms we have just considered would be the only terms on the right in equation (23). The above computation then verifies that the function u_1 given by F. John [13] is a solution of equation (20), and thus $H(z)u_1$ is a solution of equation (16) when the roots are all real, i.e., when the equation is hyperbolic.

Returning to our case, we have simplified equation (23) which now reads

$$\frac{\partial^n H(z)u_2(x,y,0^+)}{\partial z^n} = \mathrm{Re}\,\Delta^3_{\vec{x}}\,\frac{i(-1)^n}{4\pi(7-n)!}\int_{\substack{\vec{x}\cdot\vec{n}\le 0\\ 0\le\theta\le 2\pi}} d\theta\,(\vec{x}\cdot\vec{n})^{7-n}$$

$$(26)\qquad \cdot\left[\frac{-\lambda_3^n(\theta)}{Q_\lambda(\theta,-\lambda_3(\theta))} + \frac{\lambda_4^n(\theta)}{Q_\lambda(\theta,-\lambda_4(\theta))}\right] + \frac{\delta_{n,3}}{a_0}\,\delta(x)\,\delta(y)\ ,\quad n = 0,1,2,3,$$

with $\delta_{n,3} = \begin{cases} 0, & n \neq 3 \\ 1, & n = 3 \end{cases}$.

The function M may be treated in the same way to find

$$\frac{\partial^n M(x,y,0^{\pm})}{\partial z^n} = \text{Re}\, \Delta^3_{\vec{x}} \frac{(-1)^n}{(2\pi)^2(7-n)!} \int_0^{2\pi} d\theta\, [\lambda_3^n(\theta)a(\theta)+\lambda_4^n(\theta)b(\theta)] \cdot (\vec{x}\cdot\vec{n})^{7-n} \log|\vec{x}\cdot\vec{n}|$$

$$\pm\, \text{Re}\, \Delta^3_{\vec{x}} \frac{i(-1)^n}{4\pi(7-n)!} \int_{\substack{\vec{x}\cdot\vec{n}<0 \\ 0\leq\theta\leq 2\pi}} d\theta\, [-\lambda_3^n(\theta)a(\theta) + \lambda_4^n(\theta)b(\theta)](\vec{x}\cdot\vec{n})^{7-n} \tag{27}$$

$$+\, \text{Re}\, \Delta^3_{\vec{x}} \frac{c_n(-1)^n}{(2\pi)^2 7!} \int_0^{2\pi} d\theta\, [\lambda_3^n(\theta)a(\theta) + \lambda_4^n(\theta)b(\theta)](\vec{x}\cdot\vec{n})^{7-n} ,$$

$$n = 0,1,2,3.$$

We now use equations (26) and (27) to calculate the jumps of $u_3 = H(z)u_2 + M$ and its first three z derivatives across $z = 0$, $x^2 + y^2 \neq 0$. The equations that result from setting these jumps to zero are

$$\text{Re}\, \Delta^3_{\vec{x}}\, i \int_{\substack{\vec{x}\cdot\vec{n}<0 \\ 0\leq\theta\leq 2\pi}} d\theta\, (\vec{x}\cdot\vec{n})^{7-n} \left[\frac{-\lambda_3^n(\theta)}{Q_\lambda(\theta,-\lambda_3(\theta))} + \frac{\lambda_4^n(\theta)}{Q_\lambda(\theta,-\lambda_4(\theta))} + 2(-\lambda_3^n(\theta)a(\theta) + \lambda_4^n(\theta)b(\theta))\right] = 0 , \tag{28}$$

$$n = 0,1,2,3.$$

An immediate solution is

$$a(\theta) = \frac{-1}{2Q_\lambda(\theta, -\lambda_3(\theta))} \tag{29}$$

$$b(\theta) = \frac{-1}{2Q_\lambda(\theta, -\lambda_4(\theta))}$$

The proposed solution to equation (16) with $\alpha + \beta + \gamma = 1$ is then

$$u_3 = H(z)u_2 + M = \operatorname{Re} \frac{\Delta^3_{\vec{x}}}{(2\pi)^2 7!} [P(7,1)+P(7,2)+ \tfrac{1}{2}(P(7,3)+P(7,4))], \quad z > 0$$

(30)

$$u_3 = H(z)u_2 + M = -\operatorname{Re} \frac{\Delta^3_{\vec{x}}}{2(2\pi)^2 7!} [P(7,3) + P(7,4)], \quad z < 0 .$$

This proposed solution appears different from that obtained in equation (19). We now show that equation (19) and equation (30) represent the same solution of equation (16). First, the explicit derivatives $(\frac{\partial}{\partial x})^{\alpha-1} (\frac{\partial}{\partial y})^{\beta-1} (\frac{\partial}{\partial z})^{\gamma-1}$ in equation (19) occur because we solve equation (16) with $\alpha + \beta + \gamma = 5$, $\alpha, \beta, \gamma \geq 1$ while here we use $\alpha = \beta = \gamma = 1$. We must therefore check equation (30) against equation (19) only after dropping these derivatives from equation (19) (or adding them to equation (30)). Second, the functions in brackets on the right in equation (30) are five times differentiable in x, y, and z, while the corresponding terms in equation (19) are only continuous. This five-fold differentiability has been accomplished by using $\frac{1}{7!} \Delta^3_{\vec{x}} P(7,j) = P(1,j)$, $j = 1,2,3,4$. Finally, we consider the function v_1 given by

$$v_1 = \mathrm{Re}\,\frac{\Delta^3_{\vec{x}}}{(2\pi)^2 7!}\,[\sum_{j=1,2,4} P(7,j)]\;, \qquad z > 0$$

and

$$v_1 = -\mathrm{Re}\,\frac{\Delta^3_{\vec{x}}}{(2\pi)^2 7!}\,[P(7,3)]\;, \qquad z < 0\;.$$

The function v_1 is just the function v modified by removing the explicit derivatives $(\partial/\partial x)^{\alpha-1}$ $(\partial/\partial y)^{\beta-1}$ $(\partial/\partial z)^{\gamma-1}$ and by replacing P(1,j) by $\frac{1}{7!}\,\Delta^3_{\vec{x}}(7,j)$, j = 1,2,3,4. The difference between u_3 (equation (30)) and v_1 is

$$u_3 - v_1 = \mathrm{Re}\,\frac{\Delta^3_{\vec{x}}}{2(2\pi)^2\,7!}\,[P(7,3) - P(7,4)], \qquad z \neq 0\;,$$

which vanishes identically since P(7,4) - P(7,3) is pure imaginary. This may be seen by using $\lambda_4(\theta) = \overline{\lambda_3(\theta)}$, and $\overline{Q_\lambda(\theta,-\lambda_3(\theta))} = Q_\lambda(\theta,-\lambda_4(\theta))$. Our method of solution therefore gives the same answer as that found by a formal integration of the solution found by Fourier transforms.

Before we proceed to check that u_3 is a solution to equation (16) meeting all our conditions, we will show that the functions $a(\theta)$ and $b(\theta)$ given in equation (29) are the only solutions to equation (28), in the sense that any other solution $a_1(\theta)$, $b_1(\theta)$ will lead to the same solution of equation (16). This is not a uniqueness proof but a check against neglecting some other solution corresponding to different values of $a(\theta)$ and $b(\theta)$.

We begin by appling the Laplacians in equation (28) as follows: The integral in equation (28) is a sum of integrals of the form

$$G = \int\limits_{\substack{\vec{x}\cdot\vec{n}<0 \\ 0\leq\theta\leq 2\pi}} d\theta\, (\vec{x}\cdot\vec{n})^{7-n}\, g(\theta)\ , \quad g(\theta+2\pi) = g(\theta)\ , \quad n = 0,1,2,3.$$

We may write this, using polar coordinates, $r = |\vec{x}|$, $\psi = \arg \vec{x}$, as

$$G = \int_{\psi+\pi/2}^{\psi+3\pi/2} d\theta\, r^{7-n} \cos^{7-n}(\theta-\psi)\, g(\theta)\ .$$

The application of $\Delta^2_{\vec{x}} = \left(\frac{1}{r^2}\frac{\partial^2}{\partial\psi^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right)^2$ gives

$$\Delta^2_{\vec{x}} G = (7-n)(7-n-1)(7-n-2)(7-n-3) \cdot \int_{\psi+\pi/2}^{\psi+3\pi/2} d\theta\, r^{7-n-4} \cos^{7-n-4}(\theta-\psi)\, g(\theta)$$

The application of a third Laplacian will give end point contributions for $n = 2,3$ and we find

(31)
$$\Delta^3_{\vec{x}} G = (\delta_{n,0}+\delta_{n,1})\, 7! \int_{\psi+\pi/2}^{\psi+3\pi/2} d\theta\, r^{1-n}\cos^{1-n}(\theta-\psi) g(\theta) - \delta_{n,2}\frac{5!}{r^2}\left[g(\psi+\tfrac{\pi}{2})+g(\psi+\tfrac{3\pi}{2})\right] - \delta_{n,3}\frac{4!}{r^2}\left[g'(\psi+\tfrac{3\pi}{2})-g'(\psi-\tfrac{\pi}{2})\right],$$
$$n = 0,1,2,3.$$

We will need to consider equation (28) only for $n = 2,3$ to show that $a(\theta)$ and $b(\theta)$ are uniquely determined in the above sense. We use equation (31) to show equation (28), with $n = 2$, is the difference equation

$$\sin(\theta-\psi)\ [\frac{\lambda_3^2(\theta)}{Q_\lambda(\theta,-\lambda_3(\theta))} - \frac{\lambda_4^2(\theta)}{Q_\lambda(\theta,-\lambda_4(\theta))}$$

$$+ 2[\lambda_3^2(\theta)a(\theta) - \lambda_4^2(\theta)b(\theta)]]\Big|_{\theta=\psi+\pi/2}^{\theta=\psi+3\pi/2} = 0$$

Solving for $\lambda_3^2(\theta)a(\theta) - \lambda_4^2(\theta)b(\theta)$, we find the general solution of this equation as a particular solution, given by equation (29), plus the general solution of the homogeneous equation, which is any function $A(\theta)$ which satisfies

(32) $$A(\theta+\pi) = -A(\theta)$$

The functions $a(\theta)$ and $b(\theta)$ therefore satisfy

(33) $$\lambda_3^2(\theta)a(\theta) - \lambda_4^2(\theta)b(\theta) = \frac{-\lambda_3^2(\theta)}{2Q_\lambda(\theta,-\lambda_3(\theta))} - \frac{\lambda_4^2(\theta)}{2Q_\lambda(\theta,-\lambda_4(\theta))} + A(\theta)$$

For $n = 3$, we use equation (31) to show equation (28) is the differential difference equation

$$\frac{d}{d\theta}\ [\frac{\lambda_3^3(\theta)}{Q_\lambda(\theta,-\lambda_3(\theta))} - \frac{\lambda_4^3(\theta)}{Q_\lambda(\theta,-\lambda_4(\theta))}$$

$$+ 2(\lambda_3^3(\theta)a(\theta) - \lambda_4^3(\theta)b(\theta))]\ \Big|_{\theta=\psi+\pi/2}^{\theta=\psi+3\pi/2} = 0\ .$$

Solving for $\lambda_3^3(\theta)a(\theta) - \lambda_4^3(\theta)b(\theta)$, we find the general solution as a particular solution, given by equation (29), plus the general solution of the homogeneous equation, which is any function $B(\theta)$ which satisfies

(34) $$B(\theta+\pi) = B(\theta) + c, \qquad c \quad \text{constant.}$$

This gives

(35) $$\lambda_3^3(\theta)a(\theta) - \lambda_4^3(\theta)b(\theta) = \frac{-\lambda_3^3(\theta)}{2Q_\lambda(\theta,-\lambda_3(\theta))} - \frac{\lambda_4^3(\theta)}{2Q_\lambda(\theta,-\lambda_4(\theta))} + B(\theta)$$

This equation gives $B(\theta)$ in terms of functions all with period 2π, while equation (34) gives

$$B(\theta+2\pi) = B(\theta+\pi) + c = B(\theta) + 2c$$

so

$$c = 0$$

and

(36) $$B(\theta+\pi) = B(\theta) .$$

Equations (33) and (35) are linear equations for $a(\theta)$ and $b(\theta)$ with the solution

(37) $$\begin{aligned} a(\theta) &= -\frac{1}{2Q_\lambda(\theta,-\lambda_3(\theta))} + \frac{A(\theta)\lambda_4(\theta) - B(\theta)}{\lambda_3^2(\theta)(\lambda_4(\theta)-\lambda_3(\theta))} \\ b(\theta) &= -\frac{1}{2Q_\lambda(\theta,-\lambda_4(\theta))} + \frac{A(\theta)\lambda_3(\theta) - B(\theta)}{\lambda_4^2(\theta)(\lambda_4(\theta)-\lambda_3(\theta))} \end{aligned}$$

This solution is unique because the determinant of the linear system is $-2\lambda_4^2(\theta)\lambda_3^2(\theta)\ \mathrm{Im}\ (\lambda_3(\theta))$, which is never zero since $\lambda_4(\theta) = \overline{\lambda_3(\theta)}$ is not real. The particular solution we have selected in equation (29) corresponds to $A(\theta) \equiv B(\theta) \equiv 0$. If we choose other values of $A(\theta)$ and $B(\theta)$, subject to equations (32) and (36), we will find another proposed solution u_3^* different from u_3 as given in

equation (30). However, we now show that $u_3^* - u_3$ will be a polynomial of degree less than two. Hence both u_3^* and u_3 will lead to the same solution u of equation (32), because u is found by applying three or four (depending on $\alpha+\beta+\gamma$) derivatives to u_3.

To see that $u_3^* - u_3$ is a polynomial of degree less than two, we first collect those terms in the difference which contain $A(\theta)$, which gives

$$(u_3^*-u_3)_{A(\theta)} = \frac{1}{(2\pi)^2} \operatorname{Re} \int_0^{2\pi} d\theta\, A(\theta)\, 2i$$

$$\cdot \operatorname{Im}\left[\frac{\lambda_4(\theta)(\vec{x}\cdot\vec{n}-\lambda_3(\theta)z)\log(\vec{x}\cdot\vec{n}-\lambda_3(\theta)z)}{\lambda_3^2(\theta)(\lambda_4(\theta)-\lambda_3(\theta))}\right]$$

where we have applied the Laplacians. We rewrite this integral on the interval $(\pi,2\pi)$ by letting $\theta' = \theta-\pi$, and then use the relationships $A(\theta+\pi) = -A(\theta)$, $\lambda_{3,4}(\theta+\pi) = -\overline{\lambda_{3,4}(\theta)}$, and $\vec{x}\cdot\vec{n}|_{\theta+\pi} = -\vec{x}\cdot\vec{n}|_\theta$. We thus find

$$(u_3^*-u_3)_{A(\theta)} = \frac{1}{(2\pi)^2} \operatorname{Re} \int_0^{\pi} d\theta\, A(\theta)\, 2i$$

$$\cdot \operatorname{Im}\left[\operatorname{Re}\left[\frac{\lambda_4(\theta)(\vec{x}\cdot\vec{n}-\lambda_3(\theta)z)\log(\vec{x}\cdot\vec{n}-\lambda_3(\theta)z)}{\lambda_3^2(\theta)(\lambda_4(\theta)-\lambda_3(\theta))}\right] + \pi i\, \overline{\frac{\lambda_4(\theta)(\vec{x}\cdot\vec{n}-\lambda_3(\theta)z)}{\lambda_3^2(\theta)(\lambda_4(\theta)-\lambda_3(\theta))}}\right]$$

$$= \frac{1}{(2\pi)^2} \operatorname{Re} \int^{2\pi} d\theta\, A(\theta)\, 2i\pi \operatorname{Im} i\, \frac{\overline{\lambda_4(\theta)(\vec{x}\cdot\vec{n}-\lambda_3(\theta)z)}}{\lambda_3^2(\theta)(\lambda_4(\theta)-\lambda_3(\theta))}$$

which is a first degree polynomial in x, y, and z. The same method shows the terms involving B also only contribute a first degree polynomial. We have therefore not neglected any other solution to equation (16) when we chose $a(\theta)$ and $b(\theta)$ as in equation (29).

E. Verification of the Solution.

We must now show that u_3 is a solution of equation (16) with $\alpha = \beta = \gamma = 1$ and hence that

$$u \equiv \frac{\partial^{\alpha+\beta+\gamma-3} u_3}{\partial x^{\alpha-1} \partial y^{\beta-1} \partial z^{\gamma-1}} , \tag{38}$$

(where here $\alpha + \beta + \gamma = 6$ or 7) is a solution of equation (16) and, further, that u satisfies the smoothness, causality, and decay at infinity conditions given in section 5-A. We show in this subsection that u is analytic for $z < 0$ and in a neighborhood of $z = 0$, $x^2 + y^2 \neq 0$, and that u satisfies equation (16).

We begin by showing that u satisfies the causality condition. The function u_3 is defined as three Laplacians acting upon a function w_3 given by

$$w_3 = \frac{1}{(2\pi)^2 7!} \operatorname{Re} \left(P(7,1) + P(7,2) + \tfrac{1}{2}\left(P(7,3) + P(7,4)\right)\right), \quad z > 0$$

and

$$w_3 = \frac{-1}{2(2\pi)^2\, 7!} \operatorname{Re} \left(P(7,3) + P(7,4)\right) , \qquad z < 0.$$

We first show that w_3 has five continuous derivatives with respect to x, y, and z for $z > 0$ and that w_3 is analytic in x, y, and z for both $z < 0$ and in some neighborhood of the plane $z = 0$, $x^2 + y^2 + z^2 \neq 0$. We have already seen that $P(7,1) + P(7,2)$ has five continuous derivatives for $z > 0$ which all have limits as $z \to 0^+$, $x^2 + y^2 \neq 0$. The functions $P(7,3)$ and $P(7,4)$ are analytic in x, y, and z for $z \neq 0$ since their integrands are analytic in x, y and z for each $0 \leq \theta \leq 2\pi$ and these integrands are continuous in θ, x, y, and z, [14]. Moreover, $P(7,3)$ and $P(7,4)$ and all their derivatives have limits as $z \to 0^{\pm}$, $x^2 + y^2 \neq 0$, and these derivatives and limits may all be taken under the integral sign.

We now consider the function u_3 near $z = 0$, $x^2+y^2 \neq 0$. We have chosen M such that u_3 and its first three normal derivatives are continuous across $z = 0$. We will be able to apply the Cauchy-Kowalewski theorem if we can show that on $z = 0$, $\frac{\partial^n w_3}{\partial z^n}$, $n = 0,1,2,3$ are analytic in x and y, $x^2+y^2 \neq 0$. On $z = 0$, $x^2 + y^2 \neq 0$, the values of these functions may be found by taking the limit from below, i.e. as $z \to 0^-$, which by equation (27) and identity (31) gives, in polar coordinates,

$$\left.\frac{\partial^n w_3}{\partial z^n}\right|_{z=0} = \frac{\mathrm{Re}}{2(2\pi)^2(7-n)!}$$

(39)
$$\left[-r^{7-n}\log r \int_0^{2\pi} d\theta \left[\frac{(-\lambda_3(\theta))^n}{Q_\lambda(\theta,-\lambda_3(\theta))} + \frac{(-\lambda_4(\theta))^n}{Q_\lambda(\theta,-\lambda_4(\theta))}\right] + \pi i r^7 \int_{\psi+\pi/2}^{\psi+3\pi/2} d\theta \left[\frac{-(-\lambda_3(\theta))^n}{Q_\lambda(\theta,-\lambda_3(\theta))} + \frac{(-\lambda_4(\theta))^n}{Q_\lambda(\theta,-\lambda_4(\theta))}\right] \right]$$

$n=0,1,2,3.$

These functions are all analytic in r and ψ for $r \neq 0$, since the indefinite integral of an analytic function, as above, is analytic. The functions $\lambda_{3,4}(\theta)$ are analytic since they are non-coincident roots of a polynomial with coefficients analytic in θ and with highest coefficient nonzero. The plane $z = 0$ is not characteristic for our equation (this was a condition in choosing our coordinate system, and in particular, this follows from the condition that a_0 never vanish, so all roots $\lambda_i(\theta)$ are finite). We may therefore apply the Cauchy-Kowalewski theorem, which says that there is a unique analytic solution of equation (16) with $\alpha+\beta+\gamma = 1$ in the neighborhood of $z = 0$, $x^2+ y^2 \neq 0$ with data on $z = 0$, $x^2+ y^2 \neq 0$ given by three Laplacians applied to the functions given in equation (39). This then must be our solution u_3.

The above arguments show that u_3 and hence u is analytic for $z < 0$ and thus u satisfies the causality condition. We have actually shown more, i.e. that u is also analytic in a neighborhood of $z = 0$, $x^2+ y^2 \neq 0$ and that u_3 is five times continuously differentiable for $z > 0$. We will now use this information to show that u_3 is a solution of equation (16) with $\alpha = \beta = \gamma = 1$ and hence u is a solution with $\alpha+\beta+\gamma = 6$ or 7, $\alpha,\beta,\gamma \geq 1$.

We will need some preliminary results on integrals of the form

$$(40)\quad I = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx\, dy\, (\Delta^3_{\vec{x}}\, g(x,y,0)) \left[\frac{\partial^N w_3}{\partial x^a \partial y^b \partial z^c} \right]_{z=0}$$

where $0 \le N = a+b+c \le 3$, $g(x,y,z)$ is any C_0^∞ function (i.e. infinitely differentiable with compact support) and $[f]_{z=0} = f(x,y,0^+) - f(x,y,0^-)$ is the jump in f across $z = 0$. We claim all integrals of the form I are zero except when the jump function is $[\partial^3 w_3/\partial z^3]_{z=0}$ where the integral equals $\frac{1}{a_0}\, g(0,0,0)$. We may calculate the jump functions in the same manner we used for equations (26) and (27), to find

$$(41)\quad \left[\frac{\partial^N w_3}{\partial x^a \partial y^b \partial z^c}\right]_{z=0} = \mathrm{Re}\, \frac{1}{(2\pi)^2} \left[\frac{r^{7-N}\log r}{(7-N)} + c_3 r^{7-N} \right]$$

$$\cdot \int_0^{2\pi} d\theta\, \cos^a\theta\, \sin^b\theta \sum_{i=1}^{4} \frac{(-\lambda_i(\theta))^c}{Q_\lambda(\theta, -\lambda_i(\theta))}$$

which is identically zero by equation (24) for $c = 0,1,2$. The only nonvanishing jump function is then

$$\left[\frac{\partial^3 w_3}{\partial z^3}\right]_{z=0} = \frac{r^4 \log r}{a_0 4!\, 2\pi} + D\, r^4\,, \qquad D \text{ constant}$$

by equations (41) and (24).

We may now evaluate the only nonzero integral of form I, which we will call K, to be

$$K = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx\, dy\, (\Delta^3_{\vec{x}} g(x,y,0)) \left[\frac{\partial^3 w_3}{\partial z^3}\right]_{z=0}$$

$$(42) \qquad = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx\, dy\, \Delta_{\vec{x}} g(x,y,0)\left[\frac{1}{2\pi a_0}\log r + D_1\right]$$

$$= \frac{g(0,0,0)}{a_0}\ , \qquad D_1 \text{ constant.}$$

We have performed several integrations by parts and used that $(\log r)/2\pi$ is the fundamental solution of Laplace's equation in two dimensions.

The proof that u_3 is a solution of equation (32) with $\alpha = \beta = \gamma = 1$ is now straightforward. We consider the following integral M over the entire (x,y,z) space V,

$$M = \iiint_V f(x,y,z)\, Q\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) u_3(x,y,z)\, dV$$

where f is any C_0^∞ function. This, by the definition of the derivatives of a distribution, is

$$M = \iiint_V \left[\Delta^3_{\vec{x}} Q\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f(x,y,z)\right] w_3(x,y,z)\, dV.$$

We now write M as $M = M_> + M_<$ where $M_>$ is the integral over $z > 0$ and $M_<$ is over $z < 0$. We have shown that w_3 is at least five times continuously differentiable for $z \neq 0$, so we may integrate by parts repeatedly in $M_>$ and $M_<$ to cast the operator $Q(\partial/\partial z, \partial/\partial y, \partial/\partial z)$ on the function w_3. We will be left first with the volume integrals

$$N_{\gtrless} = \iiint\limits_{\substack{z>0\\ z<0}} [\Delta^3_{\vec{x}} f(x,y,z)]\, Q(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) w_3 \, dV = 0 ,$$

since w_3 is a superposition of plane waves and, therefore, a solution to the homogeneous equation for $z \neq 0$. We will also have many boundary terms. Those from infinity vanish because f has compact support. The remaining boundary terms can be grouped in pairs, one member of each pair arising from $M_>$ and the other from $M_<$, and each pair may be written in the form I, equation (40), with g as some constant times some derivative of f. These will then all be zero, by our argument above, except for those pairs which are of the form K. Such a pair can, however, arise only from the term in $Q(\partial/\partial x, \partial/\partial y, \partial/\partial z)$ proportional to $\partial^4/\partial z^4$. We then have as the only nonzero boundary term

$$a_0 \iint dx\, dy\, (\Delta^3_{\vec{x}} f(x,y,0)) [\frac{\partial^3 w_3}{\partial z^3}]_{z=0} = f(0,0,0)$$

by equation (42).

We thus have shown that

$$M = \iiint\limits_V f(x,y,z)\, Q(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})\, u_3(x,y,z)\, dV = f(0,0,0)$$

for any C_0^∞ function f. This, however, is the definition of equation (16) with $\alpha = \beta = \gamma = 1$, for u_3 a distribution. We thus have shown that u_3 is a solution of equation (16) with $\alpha = \beta = \gamma = 1$ and that $u = \partial^{\alpha+\beta+\gamma-3} u_3 / \partial^{\alpha-1}x\, \partial^{\beta-1}y\, \partial^{\gamma-1}z$ is a solution of (16) with $\alpha+\beta+\gamma = 6$ or 7, $\alpha,\beta,\gamma \geq 1$.

F. The Singularity Surfaces.

The function u has now been shown to satisfy equation (16) and the causality condition. In addition, u is homogeneous of degree -2 (-3) for $\alpha + \beta + \gamma = 6$ (7) so it will have the specified rate of decay at infinity along any path on which it is continuous. We are then left with showing that u is smooth except on the characteristic cone in $z > 0$. This will complete our proof that u is a solution to equation (16) which satisfies all the conditions set forth in Section 5-A.

We start by showing that u is smooth in all of space except possibly for those points which do not meet both of two conditions. The locus of points not satisfying both conditions is then given by geometric constructions. We next regard the characteristic cone, given by our construction (Section 3), in the space $z > 0$ as the sum of two surfaces, S_1 and S_2, and we show that those points that do not satisfy our first condition from the surface S_1 and those not satisfying the second condition form the surface S_2. The behavior of u near these surfaces will be discussed in the next section.

We have already seen that u is analytic for $z < 0$ and in a neighborhood of the plane $z = 0$, $x^2 + y^2 \neq 0$. Moreover, we also know that the functions P(7,3) and P(7,4) are analytic in x, y and z for $z \neq 0$. Therefore, the only terms in u which may be singular are, by equations (38) and

(30), the terms,

$$(43)\quad J \equiv \mathrm{Re}\ \frac{\partial^{\alpha+\beta+\gamma-3}}{\partial x^{\alpha-1}\,\partial y^{\beta-1}\,\partial z^{\gamma-1}}\ \Delta^{3}_{\vec{x}}\ \frac{1}{(2\pi)^2 7!}\,[P(7,1)+P(7,2)]$$

with $z > 0$. (We are excluding in this discussion the origin and the singularity concentrated there.)

Let us begin the description of the singularity surfaces of J by looking more closely at the function

$$(44)\quad P(7,1)+P(7,2) = \int_0^{2\pi} d\theta \sum_{i=1,2} \frac{(\vec{x}\cdot\vec{n}-\lambda_i(\theta)z)^7 \log|\vec{x}\cdot\vec{n}-\lambda_i(\theta)z|}{Q_\lambda(\theta,-\lambda_i(\theta))}$$

We consider a fixed point (x_0,y_0,z_0) and we attempt to discover whether $P(7,1) + P(7,2)$ may be differentiated infinitely often in some sufficiently small neighborhood N of $\vec{x}_0$. Differentiating repeatedly under the integral sign in equation (43) will not suffice to prove this since we have already seen that we can guarantee only five continuous derivatives of the integrand. The difficulty is the presence of the double points at $\sin\theta = 0$, where $\lambda_{1,2} = 0$. Repeated differentiations under the integral sign will also cause singularities to appear at the zeros, if any, of the functions $f_i(\vec{x},\theta) \equiv \vec{x}\cdot\vec{n}-\lambda_i(\theta)z$, $i = 1,2$, $0 \le \theta < 2\pi$. We call the roots of $f_i(\vec{x},\theta)$ for $0 \le \theta < 2\pi$ singular points and denote them by θ_{ij}, $i = 1,2$, and $j = 1,2,\ldots,m_i$.

We will be able to differentiate $P(7,1) + P(7,2)$ repeatedly after moving the contour of integration off the

real axis. To do this, we extend the logarithms into the complex plane, i.e. we replace $\log |\vec{x}\cdot\vec{n}-\lambda_i(\theta)z|$ by $\log (\vec{x}\cdot\vec{n}-\lambda_i(\theta)z)$ with suitably defined branches. This procedure will succeed if two conditions are met and the points which do not satisfy these conditions will form the surfaces of singularity for $P(7,1) + P(7,2)$ and hence for u.

The first condition arises when we define the functions $\log |f_i(\vec{x},\theta)|$, $i = 1,2$, for θ complex and for all points $\vec{x}$ in N. We define the logarithms by taking $-\pi/2 < \arg \log f_i(\vec{x}_0,\theta) \leq 3\pi/2$, $i = 1,2$, and the logarithms for all other $\vec{x} \in N$ are then given by analytic continuation. This definition assures that the logarithms coincide at the double points $\theta = 0,\pi$, for $\vec{x} = \vec{x}_0$, as they must if the integrand of $P(7,1) + P(7,2)$ is not to be singular. Our first condition is that the logarithms coincide for all points $\vec{x} \in N$ when $\theta = 0$ or π.

Before we move the θ-contour, we give some results about the functions $\lambda_{1,2}(\theta)$ and $f_{1,2}(\theta)$. The functions $\lambda_{1,2}(\theta)$ are analytic in θ in some small neighborhood of the real axis since they are roots of a polynomial with coefficients entire in θ with leading coefficient never zero. They are even analytic at the double points $\theta = 0,\pi$, as we have shown in Section 4-B. The functions $f_{1,2}(\vec{x},\theta)$ then are analytic in θ in the same neighborhood of the real axis. In particular, the number of real zeros of $f_{1,2}$ is finite in any bounded region in which $f_{1,2}$ are analytic

and thus the number of singular points is finite. We also have that if α is a root of $f_i(\vec{x},\theta)$, i = 1 or 2, which is sufficiently close to the real axis, then $\bar{\alpha}$ is also a root. To see this, we note that the power series expansions of $f_{1,2}(\vec{x},\theta)$ about any real θ have only real coefficients, since $\lambda_{1,2}(\theta)$ <u>and hence</u> $f_{1,2}(\theta)$ are real for real θ. This gives $0 = \overline{f_i(\vec{x},\alpha)} = f_i(\vec{x},\bar{\alpha})$, i = 1,2.

We now move the θ-contour into the complex plane. We first translate the interval from $0 \leq \theta \leq 2\pi$ to $-a \leq \theta \leq 2\pi - a$, $a \neq 0$, using the 2π periodicity of the integrand. The constant a is chosen so that -a and 2π-a are not singular points. They will then not belong to θ_{ij} for all $\vec{x} \in N$ since the θ_{ij} are continuous in $\vec{x}$.

This translation places all the singular points and double points in the interior of the interval of integration.

We now move the θ-contour to a new contour c which is defined as follows: c runs from -a to 2π-a on the real axis except for semicircles of radius ε in the upper half plane about each of the singular points and about $\theta = 0$ and π. We choose ε sufficiently small so that the contour remains in the region of analyticity of the integrand, the semicircles do not overlap, except obviously if $\theta_{ij} = 0$ or π for some i and j, and ε is taken to be smaller than the minimum distance from the singular points and from $\theta = 0$ to the boundary points $\theta = -a$ and $\theta = 2\pi$-a. We may make this change of contour without changing the value of

$P(7,1) + P(7,2)$ because the integrand is continuous. There will then be no contribution from the semicircles as we let their radius ε tend to zero.

We now impose our second condition which is that the singular points θ_{ij} remain real for all $\vec{x} \in N$. This will be true if the θ_{ij} are first order zeros of $f_i(\vec{x}_0, \theta)$, since, as we have just seen, complex conjugate roots must merge to form a real root.

For ε sufficiently small and $\vec{x} \in N$, we will have no singular points of the integrand of $P(7,1) + P(7,2)$ on our new contour if the above two conditions are satisfied. We may therefore differentiate $P(7,1) + P(7,2)$ infinitely often under the integral sign. The function J, and hence u, is then infinitely differentiable at the point $\vec{x}_0$.

We now give a geometric description of the locus of points which do not satisfy both of the above conditions. The surfaces formed by this locus are then the only surfaces on which u may be singular. It will be convenient to work in $\vec{\rho} = (x/z, y/z)$ space since the functions $f_{1,2}(\vec{x}, \theta)$ and the solution are homogeneous in x, y, and z. We set $f_{1,2}(\vec{x}, \theta) \equiv z(\frac{\vec{x} \cdot \vec{n}}{z} - \lambda_{1,2}(\theta)) \equiv z\, g_{1,2}(\vec{\rho}, \theta)$. The singularity surfaces only occur for $z > 0$ (neglecting, as always here, the origin), so $f_{1,2}$ and $g_{1,2}$ have the same zeros, θ_{ij}.

We begin by finding the zeros of $g_i(\vec{\rho}, \theta)$. We let $N_{1,2}$ be the normal speed loci defined in polar coordinates

by the equations $r_{1,2} = \lambda_{1,2}(\theta)$, where $r = |\vec{\rho}|$ and $\theta = \arg \vec{\rho}$. We let $\vec{\rho}\,'$ be the inverse of $\vec{\rho}$ in the unit circle and let $I_{1,2}$ be the inverse normal speed loci given by the equations $r_{1,2} = 1/\lambda_{1,2}(\theta)$. The zeros of $g_{1,2}(\vec{\rho},\theta)$, for each fixed $\vec{\rho}$, are found by considering the line $\ell_{\vec{\rho}\,'}$ which passes through $\vec{\rho}\,'$ with normal in the direction of $\vec{\rho}\,'$. If $\ell_{\vec{\rho}\,'}$ intersects $I_{1,2}$ at the point $\theta = \theta_0$, $r_{1,2} = 1/\lambda_{1,2}(\theta_0)$, then θ_0 is a root of $g_{1,2}(\vec{\rho},\theta)$ and all real roots of $g_{1,2}(\vec{\rho},\theta)$ are given in this way. This is easily seen since the line $\ell_{\vec{\rho}\,'}$ has equation $\vec{\rho}\cdot(r^2\vec{s} - \vec{\rho}) = 0$, where $\vec{s}$ is the running coordinate, and if $\vec{s} = \dfrac{1}{\lambda_i(\theta_0)}\vec{n}(\theta_0)$, $i = 1,2$, lies on $\ell_{\vec{\rho}\,'}$, then $\vec{\rho}\cdot n(\theta_0) - \lambda_i(\theta_0) = g_i(\vec{\rho},\theta_0) = 0$. This also shows that if θ_0 is a real root of $g_{1,2}(\vec{\rho},\theta)$, then $\dfrac{1}{\lambda_{1,2}(\theta_0)}\vec{n}(\theta_0)$ lies on $\ell_{\vec{\rho}\,'}$.

The second condition above is then satisfied if the singular points θ_{ij} are simple zeros of $g_{1,2}(\vec{\rho}_0,\theta)$, which is equivalent to the line $\ell_{\vec{\rho}_0'}$ having only simple intersections with $I_{1,2}$. The locus of points $\vec{\rho}$ such that $\ell_{\vec{\rho}\,'}$ is tangent to I_i form a surface which we call the i^{th} branch of the characteristic surface and denote by C_i, $i = 1,2$. These two surfaces may also be found as the envelopes of the lines $m_{i,\theta}$ with equation $\vec{n}(\theta)\cdot(\vec{s}-\lambda_i(\theta)\vec{n}(\theta)) = 0$, $i = 1,2$, with $\vec{s}$ as the running coordinate. The lines $m_{i,\theta}$ intersect the normal speed locus N_i making a right angle at the point of intersection with $\vec{n}(\theta)$. This is

the envelope of the surfaces $g_i(\vec{\rho},\theta) = 0$, $i = 1,2$. If a point $\vec{\rho}$ does not lie on one of the surfaces $C_{1,2}$, the number of real zeros of $g_{1,2}(\vec{\rho},\theta)$ is given by the number of tangents which may be drawn from $\vec{\rho}$ to $C_{1,2}$. If we consider a segment of $C_{1,2}$ which is smooth and has nonzero curvature, points sufficiently close to one side (which we call the plus side) will have two more real tangents than points sufficiently close to the other side (called the minus side). Points on the minus side will then have two complex conjugate θ_{ij} which merge into a second order singular point for points on C_i and then give two distinct real θ_{ij} for points on the plus side.

The points $\vec{\rho}$ for which the line $\ell_{\vec{\rho}'}$ passes through a point of intersection of I_1 and I_2 form a circle. In our case, where the only intersection is at infinity (when N_1 and N_2, which are inverse to I_1 and I_2 respectively, intersect at $\theta = 0,\pi$), this circle and its inverse degenerate to the line $x = 0$. The line $x = 0$ intersects the curves C_1 and C_2 at the points $\vec{\rho}_{C_i} = \left(\frac{x_{C_i}}{z_{C_i}}, \frac{y_{C_i}}{z_{C_i}}\right) = \left(0, \frac{d\lambda_i(0)}{d\theta}\right)$, since $g_i(\vec{\rho}_{C_i},0) = \frac{dg_i}{d\theta}(\vec{\rho}_{C_i},0) = 0$. We will show in the next section that the line segment in $\vec{\rho}$ space given by $x = 0$,

$$\min_{i=1,2} \frac{d\lambda_i(0)}{d\theta} \leq y \leq \max_{i=1,2} \frac{d\lambda_i(0)}{d\theta}$$

is the singularity surface corresponding to those points which do not satisfy the first condition. This, moreover, is that part of the characteristic cone given,by our construction (Section 3),

which is ruled by the tangent half lines to the line joining the slow loci of the time dependent characteristic surface. We call this line segment S_1 (it is a plane segment in (x,y,z) space) and we have that the points which don't satisfy the first condition lie on S_1.

We call the remainder of the characteristic cone S_2. It will consist just of the surfaces C_1 and C_2, since it has been shown that surfaces given by tangent half lines to the characteristic surface of the time dependent problem (excluding the plane segments caused by the presence of double points) coincide with the surfaces given by the envelope of the functions $f(\vec{x},\theta)$ with respect to θ [8].

We have now completed the proof that the function u is a solution of equation (16) which meets all the conditions set forth in Section 5-A and u may now be employed to find a solution to the steady flow problem past a source, as described earlier.

Section 6. The Solution Near the Characteristic Cone.

A. The Solution Near the Planar Segment

To investigate the solution near the characteristic cone, we first study those points which do not satisfy the first condition given in Section 5-F. This condition states that at the double points, $\theta = 0, \pi$, the two functions $\log f_i(\vec{x}, \theta)$, $i = 1,2$, should coincide for all $\vec{x}$ in a sufficiently small neighborhood, N, of the point under consideration, $\vec{x}_0$. Branches for the logarithms were given at $\vec{x} = \vec{x}_0$ and the logarithms were then defined by analytic continuation at all other $\vec{x} \in N$. We show that the points not satisfying this condition form the planar segment.

It will be helpful to rewrite $P(7,1) + P(7,2)$ in terms of integrals over $0 \le \theta \le \pi$ rather than $0 \le \theta \le 2\pi$. We have

$$(45)\quad P(7,1) + P(7,2) = \sum_{i=1,2} \int_0^{2\pi} d\theta \, \frac{(\vec{x}\cdot\vec{n} - \lambda_i(\theta)z)^7 \log|\vec{x}\cdot\vec{n} - \lambda_i(\theta)z|}{Q_\lambda(\theta, -\lambda_i(\theta))}$$

$$= \sum_{i=1,2} 2\int_0^{\pi} \frac{(\vec{x}\cdot\vec{n} - \lambda_i(\theta)z)^7 \log|\vec{x}\cdot\vec{n} - \lambda_i(\theta)z|}{Q_\lambda(\theta, -\lambda_i(\theta))}$$

since $\lambda_i(\theta+\pi) = -\lambda_i(\theta)$ and $Q_\lambda(\theta, -\lambda_i(\theta+\pi)) = -Q_\lambda(\theta, -\lambda_i(\theta))$, by equations (10) and (15) respectively. As in Section 5-F, we extend the logarithms and move the contour to c' which is that part of c with $-a \le \theta \le \pi - a$. The only double point is now at $\theta = 0$ where $\lambda_{1,2}$ vanish.

In this section, we will always have $i = 1,2$. We expand $f_i(\vec{x},\theta) = \vec{x}\cdot\vec{n} - \lambda_i(\theta)z$ in a Taylor series about $\theta = 0$ and $\vec{x}=\vec{x}_0$. Ignoring second order terms, we find

$$(46) \quad f_i(\vec{x},\theta) = x_0 + \frac{d}{d\theta}(\vec{x}\cdot\vec{n} - \lambda_i(\theta)z)\Big|_{\theta=0}\theta + (x-x_0) + \ldots$$

or

$$f_i(\vec{x},\theta) = x + (y_0 + \frac{d\lambda_i(0)}{d\theta} z_0)\theta + \ldots$$

For x sufficiently close to $x_0 \neq 0$ and θ sufficiently small, we have sgn Re f_i = sgn x = sgn x_0. Therefore the functions f_i, which are the arguments of the logarithms, do not cross the imaginary axis for $\vec{x} \in N$ and θ sufficiently small. The logarithms will coincide at $\theta = 0$ for all $\vec{x} \in N$ if they do so at $\vec{x} = \vec{x}_0$, and the first condition is satisfied.

At $x_0 = 0$, the first condition will still be satisfied if the logarithms still coincide at $\theta = 0$ after analytic continuation across the plane $x = 0$. Near $\theta = 0$, c' consists of a semicircle of radius ε in the upper half plane so that Im $\theta \geq 0$. We consider f_i for $\vec{x} = \vec{x}_1 \in N$ and let $\vec{x}_1$ have positive x-coordinate. Equation (46) shows that Re $f_i(\vec{x}_1,\theta) > 0$ and sgn (Im $f_i(\vec{x}_1,\theta)$) = sgn$(y_0 - (d\lambda_i(0)/d\theta)\, z_0)$. (We do not consider here the intersections of the planar segment with C_i, i.e. the points $x_0 = 0$, $y_0/z_0 = d\lambda_i(0)/d\theta$).

If we now analytically continue log f_i across the plane $x = 0$ to a point $\vec{x}_2$, equation (46) shows the real part of f_i is negative and we again have

$\operatorname{sgn}(\operatorname{Im} f_i(\vec{x}_2,\theta)) = \operatorname{sgn}(y_0 - \frac{d\lambda_i(0)}{d\theta} z_0)$. We now let θ tend to zero and find $\operatorname{Im}\log f_i(\vec{x}_2,0) = \operatorname{sgn}(y_0 - \frac{d\lambda_i(0)}{d\theta} z_0)\pi$. Let $y_{0,max} = \max_i (\frac{d\lambda_i(0)}{d\theta} z_0)$ and $y_{0,min} = \min_i (\frac{d\lambda_i(0)}{d\theta} z_0)$. If $y < y_{0,min}$ or $y > y_{0,max}$, the logarithms will still coincide at $\theta = 0$ and the first condition will be satisfied.

On the other hand, for $y_{0,min} < y < y_{0,max}$, let $(P(7,1) + P(7,2))_A$ be the analytic continuation of $P(7,1) + P(7,2)$ across the plane $x = 0$. We calculate the jump across this plane obtaining

$$(P(7,1) + P(7,2)) - (P(7,1) + P(7,2))_A$$

$$= - \operatorname{Re}\, 2\pi i H(-x) \sum_{j=1,2} \int_{c'} d\theta \frac{(\vec{x}\cdot\vec{n} - \lambda_j(\theta) z)^7 \operatorname{sgn}(y_0 - \frac{d\lambda_j(0)}{d\theta} z_0)}{Q_\lambda(\theta, -\lambda_j(\theta))}$$

The above integral is evaluated by moving c' onto the real axis yielding $-\pi i$ times the residue at $\theta = 0$ and a principal value integral, which does not contribute since it is real. We then have

$$(P(7,1) + P(7,2)) - (P(7,1) + P(7,2))_A$$

$$= \frac{2\pi^2 H(-x)(-x)^7}{(\frac{d\lambda_2(0)}{d\theta} - \frac{d\lambda_1(0)}{d\theta})|\lambda_3(0)|^2} [\operatorname{sgn}(y_0 - \frac{d\lambda_1(0)}{d\theta} z_0) + \operatorname{sgn}(y_0 - \frac{d\lambda_2(0)}{d\theta} z_0)]$$

$$= \frac{-4\pi^2 H(-x)(-x)^7}{|\frac{d\lambda_2(0)}{d\theta} - \frac{d\lambda_1(0)}{d\theta}|\ |\lambda_3(0)|^2}$$

The jump across $x = 0$ in the solution to equation (16) is then

$$(47)\quad u - u_A = \frac{\partial^{\alpha+\beta+\gamma-3}}{\partial x^{\alpha-1}\partial y^{\beta-1}\partial z^{\gamma-1}} \frac{-H(-x)(-x)}{\left|\frac{d\lambda_2(0)}{d\theta} - \frac{d\lambda_1(0)}{d\theta}\right| |\lambda_3(0)|^2}$$

for $y_{0,min} < y_0 < y_{0,max}$ and $z_0 > 0$.

The planar segment is then given by $x = 0$, $y_{0,min} < y < y_{0,max}$, and $z > 0$.

We will now calculate the values of $\frac{d\lambda_i(0)}{d\theta}$ in the two coordinate systems given earlier (Section 4).

For the system with z-axis perpendicular to the direction of $\vec{A}_0$, the values of $d\lambda_i(0)/d\theta$ are calculated from $Q(\cos\theta, \sin\theta, \lambda)$ (equation (11). We note that Q is independent of $\cos\theta$.

Let $u = \sin\theta$, then in (u,λ)-space the line given by $\lambda = pt$, $u = qt$, $-\infty < t < \infty$ is tangent to $\lambda = -\lambda_i(u)$ at the origin if the direction numbers (p,q) of the line satisfy

$$Q_{\lambda\lambda}(0,0)p^2 + 2\,Q_{\lambda u}(0,0)pq + Q_{uu}(0,0)q^2 = 0\ ,$$

where

$$Q_{\lambda\lambda}(0,0) = -\,u_0^4\left(\frac{1}{m^2} + \frac{1}{M^2}\right) 2\sin^2\phi\ ,$$

$$Q_{\lambda u}(0,0) = u_0^4\left(\frac{1}{m^2} + \frac{1}{M^2}\right) 2\sin\phi\cos\phi\ ,$$

$$Q_{uu}(0,0) = -2u_0^4\left(\frac{1}{m^2} + \frac{1}{M^2}\right)\cos^2\phi + \frac{2}{m^2M^2}\ .$$

Since $\frac{d\lambda_i}{du} = \frac{d\lambda_i}{d\theta}\frac{1}{\cos\theta}$, we have

$$\left.\frac{d\lambda_i}{du_\pm}\right|_{u=0} = \left.\frac{d\lambda_i}{d\theta}\right|_{\theta=0} = -\cot\phi \pm \frac{1}{\sqrt{(m^2+M^2)\sin^2\phi}}$$

yielding two distinct tangents.

In the second coordinate system, the same method gives

$$\left(\frac{p}{q}\right)_\pm = \frac{m^2M^2\cos\phi\sin\phi \pm \sqrt{(m^2+M^2)m^2M^2\sin^2\phi}}{(m^2+M^2) - m^2M^2\cos^2\phi} .$$

This coordinate system is only used, as noted in Section 4-B, when $1-(m^2+M^2) + m^2M^2\cos^2\phi < 0$. This implies that the denominator above is greater than one, hence again giving two distinct tangents.

B. Steady Flow Across the Planar Segment.

The solutions of equation (6) are found by forming the appropriate linear combinations of u for various values of α, β, γ, with $\alpha+\beta+\gamma = 6$ or 7. Equation (47) shows that unless $\beta = \gamma = 1$, the jump $u-u_A$ across the planar segment is zero. We now show that $\beta > 1$ and/or $\gamma > 1$ for every term of these linear combinations, i.e. each term is differentiated at least once with respect to either y or z. Thus the steady flow solution with all its derivatives is continuous across the planar segment of the characteristic cone.

For the second coordinate system, $\vec{A}_0 = A_0(0,\sin\phi,\cos\phi) = A_0\vec{n}_0$, and $\vec{u}_0 = u_0(0,0,1)$.

The equation for $v_1 = \rho$ (with $\Delta = D_x^2 + D_y^2 + D_z^2$) is

$$u_0[D_z^4 - (\frac{1}{m^2} + \frac{1}{M^2})\Delta D_z^2 + \frac{1}{m^2M^2} (\vec{n}_0 \cdot \nabla)^2]\rho$$

$$= [D_z^2 - \frac{1}{M^2} \Delta]D_z\rho_s + \frac{1}{M^2} \Delta D_z(\vec{n}_0 \cdot \vec{B}_s) + \frac{1}{M^2} \Delta(\vec{n}_0 \cdot \nabla)(\vec{n}_0 \cdot \vec{M}_s).$$

Every term on the right side is differentiated at least once with respect to either y or z giving the result for $v_1 = \rho$.

Equation (5A) gives $v_2 = \alpha$ in terms of ρ_z and ρ_s, so α is also smooth across the planar segment. The equation satisfied by $v_3 = \beta$ is

$$[D_z^4 - (\frac{1}{m^2} + \frac{1}{M^2})\Delta D_z^2 + \frac{1}{m^2M^2} \Delta(\vec{n}_0 \cdot \nabla)^2]\beta$$

$$= [- \frac{1}{m^2M} D_z^2 + \frac{\Delta}{m^2M^3}] (\vec{n}_0 \cdot \nabla)\rho_s$$

$$+ [\frac{1}{M} D_z^3 - \frac{\Delta}{M(M^2+m^2)}] (\vec{n}_0 \cdot \vec{M}_s) - \frac{1}{M^3m^2} \Delta(\vec{n}_0 \cdot \nabla) (\vec{n}_0 \cdot \vec{B}_s)$$

and this gives the result for β because again all the source terms are differentiated at least once with respect to either y or z.

Finally, equation (5B) may be written in the form

$$A_0\gamma_{xx} = u_0 \nabla \cdot \vec{M}_s - u_0^2\alpha_z - a_0^2\Delta\rho - A_0(\gamma_{yy} + \gamma_{zz}) .$$

The terms on the right side are all smooth across the planar segment because we have shown that α and ρ are smooth and differentiation of $v_4 = \gamma$ with respect to z or y removes any possible discontinuity of γ. This implies that the function γ_{xx} is smooth, which is possible only if γ is smooth. Therefore, the planar segment for flow past a source is not a surface of singularity.

C: The Solution Near the Remainder of the Characteristic Cone.

The terms in the solution which are singular on the characteristic cone are those which are superpositions of plane wave functions with real wave speeds. These terms form the function J, (equation (43)). They are similar to the terms which comprise the plane wave representation of the fundamental solution of the Cauchy problem for a hyperbolic equation. The behavior of the fundamental solution near the characteristic cone has been treated in the hyperbolic case [3]. Our discussion will therefore be brief and will serve chiefly to show the similarity with the hyperbolic case.

We consider the solution in the neighborhood of a point $\vec{\rho}_0 = (x_0/z_0, y_0/z_0)$ which lies in the interior of a smooth segment S of C_i. We assume S does not intersect any other singularity surface and that the curvature of C_i at $\vec{\rho}_0$ is not zero. As noted in Section 5-F, it is possible to draw two more tangents to S from points sufficiently close

to one side of S, the plus side, than is possible for points near the other side, the minus side.

We rewrite J in terms of integrals over $0 \leq \theta \leq \pi$ rather than $0 \leq \theta \leq 2\pi$, using equation (45). Since $\vec{\rho}_0 \in S$, we have $g_i(\vec{\rho}_0,\theta_0) = g_{i,\theta}(\vec{\rho}_0,\theta_0) = 0$ for some $\theta_0 \in [0,\pi]$. There are two complex conjugate roots of $g_i(\vec{\rho},\theta)$ near $\vec{\rho} = \vec{\rho}_0$ and $\theta = \theta_0$, for $\vec{\rho}$ on the minus side. These merge on the real axis at $\theta = \theta_0$ when $\vec{\rho} = \vec{\rho}_0$, and give two real singular points for $\vec{\rho}$ on the plus side.

We first consider a point $\vec{\rho}$ on the plus side and near $\vec{\rho}_0$. In a neighborhood of θ_0, there will be two real singular points, α_1 and α_2. We move the contour from the real axis to the contour c', as in Section 6-A. The contour c' lies on the real axis for θ near θ_0, except for semicircles in the upper half plane of radius ε about $\theta = \alpha_1$ and $\theta = \alpha_2$. For $\vec{\rho}$ sufficiently close to $\vec{\rho}_0$, we may move c' to a contour having one semicircle in the upper half plane which avoids the points θ_0, α_1 and α_2, instead of the two (smaller) semicircles. Furthermore, this semicircle may be so chosen that it avoids these three points as $\vec{\rho} \to \vec{\rho}_0$. This follows from the continuity of the singular points in $\vec{\rho}$. We thus find the solution and all its derivatives remain finite as $\vec{\rho} \to \vec{\rho}_0$ from the plus side.

We now let $\vec{\rho}$ be a point near $\vec{\rho}_0$ on the minus side. There are two complex conjugate roots, β_1 and β_2, of $g_i(\vec{\rho},\theta)$ near $\theta = \theta_0$, which merge on the real axis at $\theta = \theta_0$

for $\vec{\rho} = \vec{\rho}_0$. We move to the contour c'. The contour c' avoids all other singular points and the double point $\theta = 0$ in a neighborhood of $\vec{\rho}_0$, except for β_1 and β_2. We are using that S intersects no other singular surface. The only singularity of the solution as $\vec{\rho} \to \vec{\rho}_0$ will then come from the integration over a small neighborhood, $N(\theta_0)$, of θ_0 on the real axis. The term in J which will be singular as $\vec{\rho} \to \vec{\rho}_0$ is

$$I_i = \frac{1}{2\pi^2} \frac{\partial^{\alpha+\beta+\gamma-3}}{\partial x^{\alpha-1}\, \partial y^{\beta-1}\, \partial z^{\gamma-1}} \int_{N(\theta_0)} dz\, \frac{z\, g_i(\vec{\rho},\theta) \log (z g_i(\vec{\rho},\theta))}{Q_\lambda(\theta, -\lambda_i(\theta))},$$

$$\alpha + \beta + \gamma = 6 \text{ or } 7,$$

where we have applied the Laplacians under the integral sign.

We assume $\alpha + \beta + \gamma = 6$ and apply the explicit derivatives under the integral sign. The behavior of other derivatives of I_i may be found exactly as below.

We obtain

$$I_i = \frac{(-1)^{\gamma-1}}{2\pi^2 z^2} \int_{N(\theta_0)} d\theta\, \frac{\cos^{\alpha-1}\theta\, \sin^{\beta-1}\theta\, \lambda_i^{\gamma-1}(\theta)}{g_i^2(\vec{\rho},\theta)\, Q_\lambda(\theta, -\lambda_i(\theta))}$$

We expand $g_i(\vec{\rho},\theta)$ about $(\vec{\rho}_0,\theta_0)$, ignoring terms of third order, to find

$$g_i(\vec{\rho},\theta) = \vec{n}(\theta_0)\cdot(\vec{\rho} - \vec{\rho}_0) + \vec{n}_{,\theta}(\theta_0) \cdot (\vec{\rho} - \vec{\rho}_0)(\theta-\theta_0) - \tfrac{1}{2}R(\theta-\theta_0)^2 + \ldots$$

with $R = \vec{n}(\theta_0) \cdot (\vec{\rho} - \vec{\rho}_0) + \lambda_{i,\theta\theta}(\theta_0)$.

We then have

$$\frac{1}{g_i^2(\vec{\rho},\theta)} = \frac{1}{[\vec{n}(\theta_0)\cdot(\vec{\rho}-\vec{\rho}_0)+\vec{n}_{,\theta}(\theta_0)\cdot(\vec{\rho}-\vec{\rho}_0)(\theta-\theta_0)-\frac{1}{2}R(\theta-\theta_0)^2]^2} +F,$$

where the integral of F remains bounded on $N(\theta_0)$ as $\vec{\rho} \to \vec{\rho}_0$. Neglecting terms which are finite as $\vec{\rho} \to \vec{\rho}_0$, we may approximate I_i by

$$(48)\quad I_i \sim \frac{(-1)^{\gamma-1}}{2\pi^2 z^2} \frac{\cos^{\alpha-1}\theta_0 \sin^{\beta-1}\theta_0 \lambda_i^{\gamma-1}(\theta_0)}{Q_\lambda(\theta_0, -\lambda_i(\theta_0))}$$

$$\cdot \int_{N(\theta_0)} \frac{d\theta}{[\vec{n}(\theta_0)\cdot(\vec{\rho}-\vec{\rho}_0)+\vec{n}_{,\theta}(\theta_0)\cdot(\vec{\rho}-\vec{\rho}_0)(\theta-\theta_0)-\frac{1}{2}R(\theta-\theta_0)^2]^2}$$

The vector $\vec{n}(\theta_0)$ is a unit normal to S at $\vec{\rho}_0$, since C_i was constructed (Section 5-F) as the envelope of the lines $m_{i,\theta}$ which intersect the normal speed locus and are perpendicular to the vector from the origin at the point of intersection. The tangent line to C_i at $\vec{\rho} = \vec{\rho}_0$ then has $\vec{n}(\theta_0)$ as a unit normal, and $|\vec{n}(\theta_0)\cdot(\vec{\rho}-\vec{\rho}_0)|$ is the perpendicular distance from $\vec{\rho}$ to $\vec{\rho}_0$. We note that this distance cannot be zero for $\vec{\rho} \neq \vec{\rho}_0$, since $\vec{\rho}$ is near $\vec{\rho}_0$ and lies on the minus side of S.

We now show that the constant $|R|$ is the radius of curvature of C_i at $\vec{\rho}_0$. We have $g_i(\vec{\rho}_0,\theta_0) = \vec{\rho}_0\cdot \vec{n}(\theta_0)-\lambda_i(\theta_0) = 0$, so $R = \lambda_{i,\theta\theta}(\theta_0) + \lambda_i(\theta_0)$. The curve C_i is the envelope of the curves $g_i(\vec{\rho},\theta) = 0$ with respect to θ. This gives the

simultaneous equations $g_i(\vec{\rho},\theta) = g_{i,\theta}(\vec{\rho},\theta) = 0$, which may be easily solved to yield

$$\vec{\rho} = \lambda_i(\theta)\, \vec{n}(\theta) + \lambda_{i,\theta}\, \vec{n}_{,\theta}(\theta) \;, \qquad 0 \leq \theta < \pi \;,$$

as an equation for C_i. We then have

$$\kappa\, \vec{N}(\theta) = - \frac{R\, \vec{n}(\theta)}{(\lambda+\lambda_{,\theta\theta})^2} = - \frac{\vec{n}(\theta)}{R}$$

where $\kappa > 0$ is the curvature and $\vec{N}$ is the unit normal in the minus direction, giving the result. We note that $- R\, \vec{n}(\theta_0) \cdot (\vec{\rho} - \vec{\rho}_0) > 0$, independent of the relative directions of $\vec{n}(\theta_0)$ and $\vec{N}(\theta_0)$.

We may now evaluate the integral in equation (48) and, keeping only the highest order singularity as $\vec{\rho} \to \vec{\rho}_0$, we find

$$I_i \sim \frac{(-1)^{\gamma-1}}{2\pi^2 z^2} \; \frac{\cos^{\alpha-1}\theta_0 \, \sin^{\beta-1}\theta_0 \, \lambda_i^{\gamma-1}(\theta_0)\, 2\pi \, \mathrm{sgn}(R) R}{Q_\lambda(\theta_0, -\lambda_i(\theta_0))\, (-2R\vec{n}\cdot(\vec{\rho} - \vec{\rho}_0))^{3/2}} + \ldots \; .$$

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$$A_0 < a_0$$

$$\overrightarrow{OP} = -\overrightarrow{OP}' = \overrightarrow{A}_0$$

$$OQ = OQ' = a_0$$

$$OR = OR' = \sqrt{\frac{a_0^2 \, A_0^2}{a_0^2 + A_0^2}}$$

$$OS = OS' = \sqrt{a_0^2 + A_0^2}$$

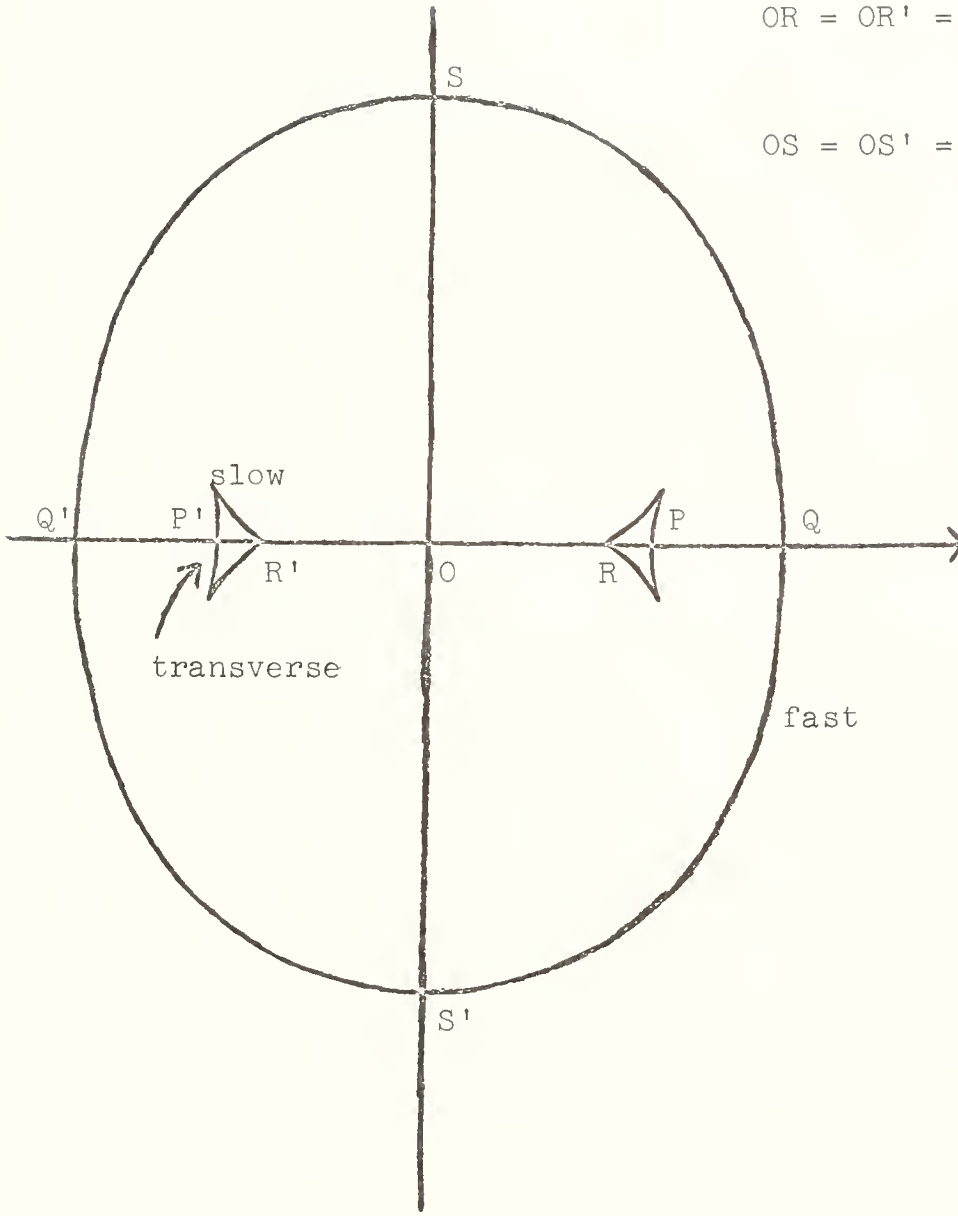

Figure 1. Section of the characteristic surface containing $\overrightarrow{A}_0$.

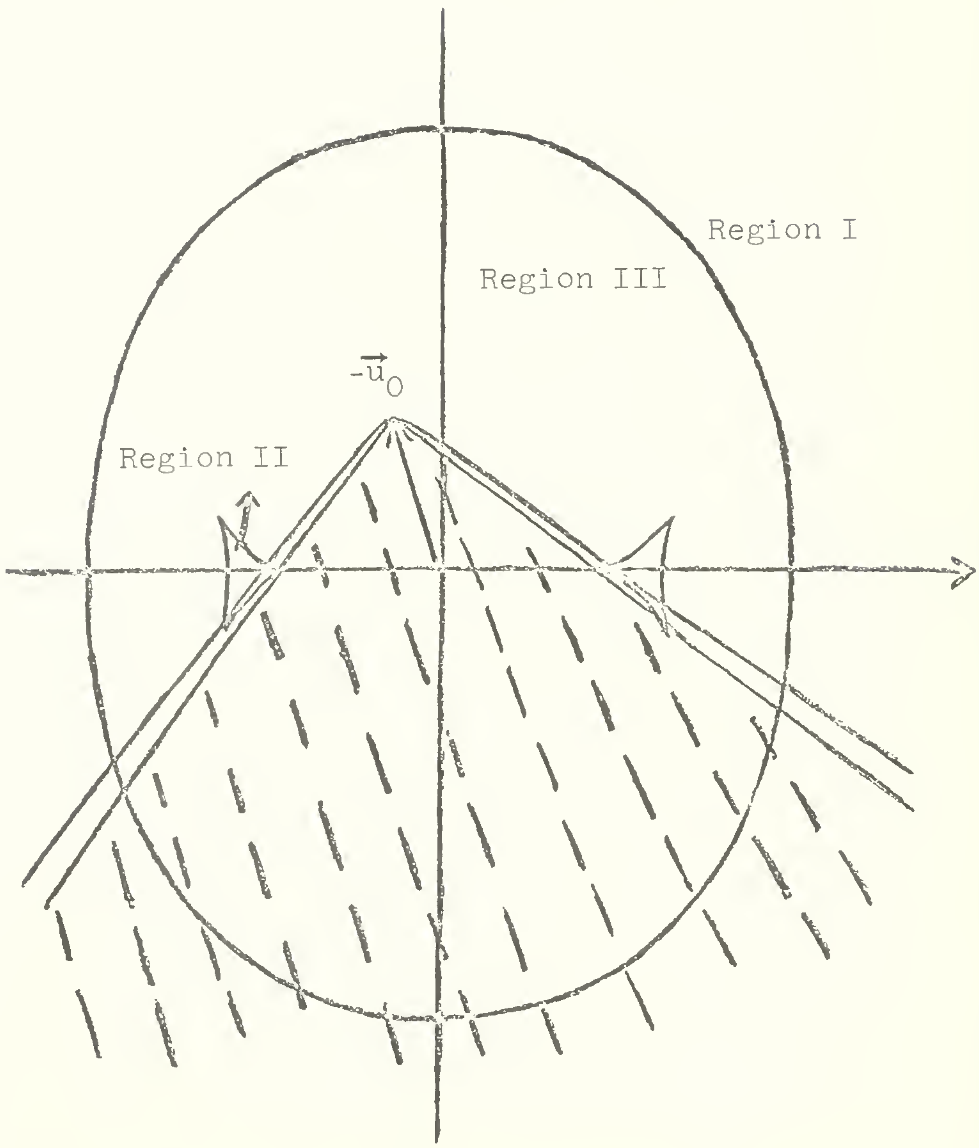

Figure 2. Intersection of the section of the characteristic surface containing $\vec{A}_0$ and the tangent cones from $-\vec{u}_0$. The shaded region is the planar segment.

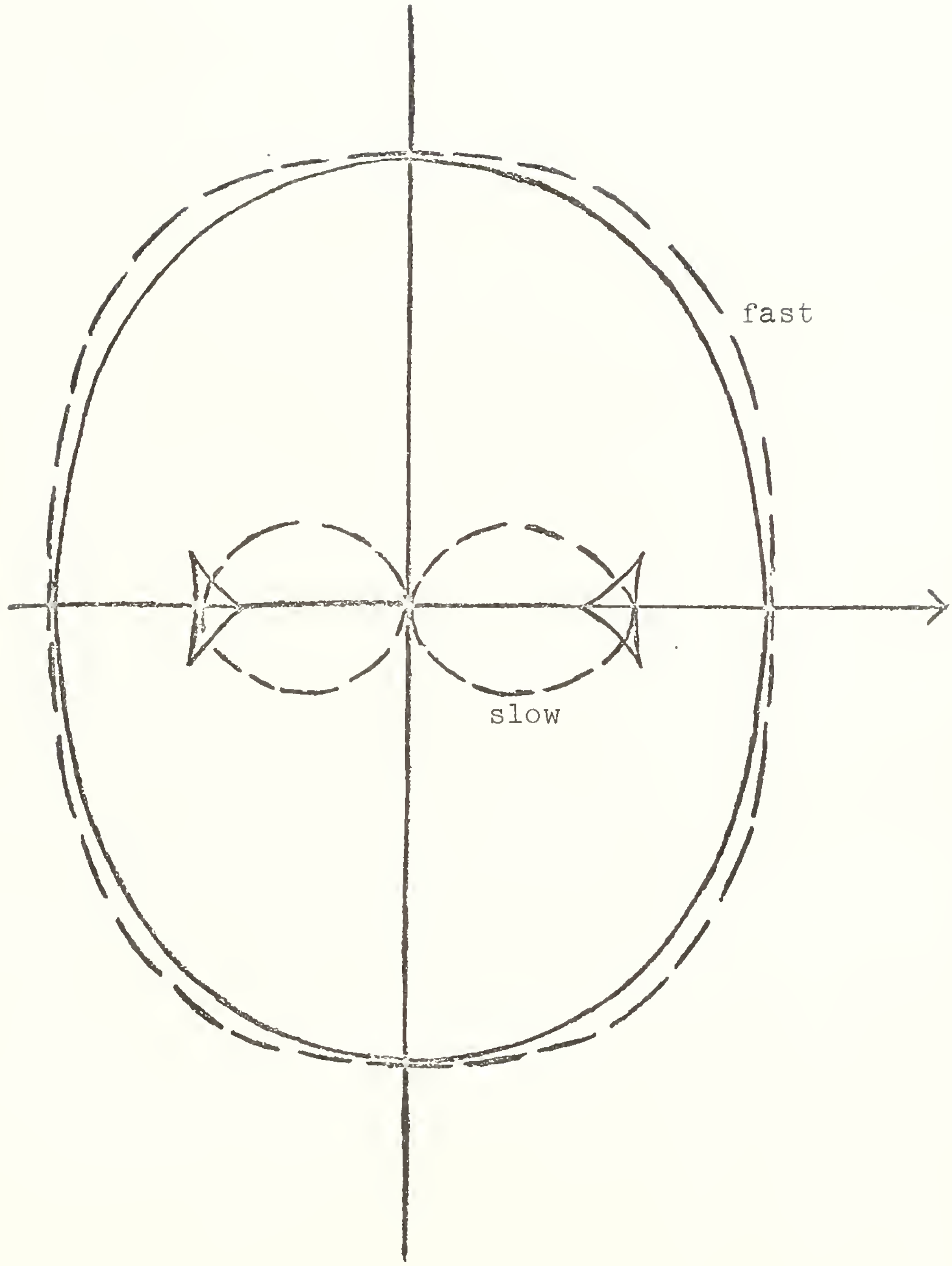

Figure 3. Intersection of the section of the characteristic surface containing $\vec{A}_0$ (solid lines) and the normal speed locus (dashed lines).